

GPGCD: An iterative method for calculating approximate GCD of univariate polynomials[☆]

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Abstract

We present an iterative algorithm for calculating approximate greatest common divisor (GCD) of univariate polynomials with the real or the complex coefficients. For a given pair of polynomials and a degree, our algorithm finds a pair of polynomials which has a GCD of the given degree and whose coefficients are perturbed from those in the original inputs, making the perturbations as small as possible, along with the GCD. The problem of approximate GCD is transferred to a constrained minimization problem, then solved with the so-called modified Newton method, which is a generalization of the gradient-projection method, by searching the solution iteratively. We demonstrate that, in some test cases, our algorithm calculates approximate GCD with perturbations as small as those calculated by a method based on the structured total least norm (STLN) method and the UVGCD method, while our method runs significantly faster than theirs by approximately up to 30 or 10 times, respectively, compared with their implementation. We also show that our algorithm properly handles some ill-conditioned polynomials which have a GCD with small or large leading coefficient.

Keywords: Approximate polynomial GCD, Gradient-projection method, Ill-conditioned problem, Optimization.

1. Introduction

For algebraic computations on polynomials and matrices, approximate algebraic algorithms are attracting more attention than before. These algorithms take inputs with some “noise” such as polynomials with floating-point number coefficients with rounding errors, or more practical errors such as measurement

[☆]Preliminary versions of this paper have been presented at ISSAC’09 (Seoul, Republic of Korea, July 28–31, 2009) [26] and The Joint Conference of ASCM-MACIS 2009 (Fukuoka, Japan, December 14–17, 2009) [25].

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errors, then, with minimal changes on the inputs, seek a meaningful answer that reflect desired property of the input, such as a common factor of a given degree. By this characteristic, approximate algebraic algorithms are expected to be applicable to more wide range of problems, especially those to which exact algebraic algorithms were not applicable.

As an approximate algebraic algorithm, we consider calculating the approximate greatest common divisor (GCD) of univariate polynomials with the real or the complex coefficients, such that, for a given pair of polynomials and a degree d , finding a pair of polynomials which has a GCD of degree d and whose coefficients are perturbations from those in the original inputs, while making the perturbations as small as possible, along with the GCD. This problem has been extensively studied with various approaches including the Euclidean method on the polynomial remainder sequence (PRS) ([1], [22], [23]), the singular value decomposition (SVD) of the Sylvester matrix ([7], [10]), the LU or QR factorization of the Sylvester and/or Bézout matrix or their displacements ([2], [4], [8], [29], [32])¹, Padé approximation ([19]), optimization strategies ([5], [6], [13], [14], [15], [31]). Furthermore, stable methods for ill-conditioned problems have been discussed ([2], [8], [17], [21]).

Among methods in the above, we focus our attention on optimization strategy in this paper, especially iterative method for approaching an optimal solution, after transferring the approximate GCD problem into a constrained minimization problem. Already proposed algorithms utilize iterative methods including the Levenberg-Marquardt method ([6]), the Gauss-Newton method ([31]) and the structured total least norm (STLN) method ([13], [14]). Among them, STLN-based methods have shown good performance calculating approximate GCD with sufficiently small perturbations efficiently.

Here, we utilize the so-called modified Newton method ([24]), which is a generalization of the gradient-projection method ([20]), for solving the constrained minimization problem. This method has interesting features such that it combines the *projection* and the *restoration* steps in the original gradient-projection method, which reduces the number of solving a linear system. We demonstrate that our algorithm calculates approximate GCD with perturbations as small as those calculated by the STLN-based methods, while our method show significantly better performance over them in its speed compared with their implementation, by approximately up to 30 times. Furthermore, we also show that our algorithm can properly handle some ill-conditioned problems such as those with GCD containing small or large leading coefficient. We call our algorithm *GPGCD* after the initials of the *gradient projection* method.

In this paper, we present the following expansion from the previous results ([26], [25]) as presenting the new algorithm for monic polynomials to calculate perturbed polynomials without giving perturbations for the leading coefficients;

¹Note that the article by Bini and Boito [2] has absence of reference to a literature on computation on structured matrices by Pan [18], whereas the dissertation by Boito [4] has no such omission.

providing experiment results for new test polynomials that have been prepared more carefully and comparison with the UVGCD method ([30]) (in Section 5.2); adding more experiments for comparison of our algorithm with the STLN-based method and the UVGCD method (in Section 5.3).

The rest part of this chapter is organized as follows. In Section 2, we transform the approximate GCD problem into a constrained minimization problem. In Section 3, we review the framework of the gradient-projection method and the modified Newton method. In Section 4, we show an algorithm for calculating the approximate GCD, and discuss issues in the application of the gradient-projection method or the modified Newton method. In Section 5, we demonstrate performance of our algorithm with experiments.

2. Formulation of the Approximate GCD Problem

Let $F(x)$ and $G(x)$ be univariate polynomials with the real or the complex coefficients, given as

$$\begin{aligned} F(x) &= f_m x^m + f_{m-1} x^{m-1} + \cdots + f_0, \\ G(x) &= g_n x^n + g_{n-1} x^{n-1} + \cdots + g_0, \end{aligned} \tag{1}$$

with $0 < n \leq m$. We permit F and G to be relatively prime in general. For a given integer d satisfying $0 < d \leq n$, let us calculate a deformation of $F(x)$ and $G(x)$ in the form of

$$\begin{aligned} \tilde{F}(x) &= F(x) + \Delta F(x) = H(x) \cdot \bar{F}(x), \\ \tilde{G}(x) &= G(x) + \Delta G(x) = H(x) \cdot \bar{G}(x), \end{aligned} \tag{2}$$

where $\Delta F(x)$, $\Delta G(x)$ are polynomials whose degrees do not exceed those of $F(x)$ and $G(x)$, respectively, $H(x)$ is a polynomial of degree d , and $\bar{F}(x)$ and $\bar{G}(x)$ are pairwise relatively prime. If we find \tilde{F} , \tilde{G} , \bar{F} , \bar{G} and H satisfying (2), then we call H an *approximate GCD of F and G* . For a given degree d , we tackle the problem of finding an approximate GCD H while minimizing the norm of the deformations $\|\Delta F(x)\|_2^2 + \|\Delta G(x)\|_2^2$.

To make the paper self-contained, we define notations in the theory of sub-resultants used below.

Definition 1 (Sylvester Matrix). Let F and G be defined as in (1). The *Sylvester matrix* of F and G , denoted by $N(F, G)$, is an $(m+n) \times (m+n)$ matrix constructed from the coefficients of F and G , such that

$$N(F, G) = \begin{pmatrix} f_m & & & g_n & & \\ \vdots & \ddots & & \vdots & \ddots & \\ f_0 & & f_m & g_0 & & g_n \\ & \ddots & \vdots & & \ddots & \vdots \\ & & f_0 & & g_0 & \end{pmatrix}.$$

$\underbrace{\hspace{10em}}_n \qquad \underbrace{\hspace{10em}}_m$

Definition 2 (Subresultant Matrix). Let F and G be defined as in (1). For $0 \leq j < n$, the j -th subresultant matrix of F and G , denoted by $N_j(F, G)$, is an $(m+n-j) \times (m+n-2j)$ sub-matrix of $N(F, G)$ obtained by taking the left $n-j$ columns of coefficients of F and the left $m-j$ columns of coefficients of G , such that

$$N_j(F, G) = \begin{pmatrix} f_m & & g_n & & \\ \vdots & \ddots & \vdots & \ddots & \\ f_0 & & f_m & g_0 & g_n \\ & \ddots & \vdots & \ddots & \vdots \\ & & f_0 & & g_0 \end{pmatrix}. \quad (3)$$

$\underbrace{\hspace{10em}}_{n-j} \qquad \underbrace{\hspace{10em}}_{m-j}$

Definition 3 (Subresultant). Let F and G be defined as in (1). For $0 \leq j < n$ and $k = 0, \dots, j$, let $N_{j,k} = N_{j,k}(F, G)$ be a sub-matrix of $N_j(F, G)$ obtained by taking the top $m+n-2j-1$ rows and the $(m+n-j-k)$ -th row (note that $N_{j,k}(F, G)$ is a square matrix). Then, the polynomial

$$S_j(F, G) = |N_{j,j}|x^j + \dots + |N_{j,0}|x^0$$

is called the j -th subresultant of F and G .

Now, in the case $\tilde{F}(x)$ and $\tilde{G}(x)$ have a GCD of degree d , then the theory of subresultants tells us that the $(d-1)$ -th subresultant of \tilde{F} and \tilde{G} becomes zero, namely we have

$$S_{d-1}(\tilde{F}, \tilde{G}) = 0.$$

Then, the $(d-1)$ -th subresultant matrix $N_{d-1}(\tilde{F}, \tilde{G})$ has a kernel of dimension equal to 1. Thus, there exist polynomials $A(x), B(x) \in \mathbf{R}[x]$ or $\mathbf{C}[x]$ satisfying

$$A\tilde{F} + B\tilde{G} = 0, \quad (4)$$

with $\deg(A) < n-d$ and $\deg(B) < m-d$ and $A(x)$ and $B(x)$ are relatively prime. Therefore, for the given $F(x)$, $G(x)$ and d , our problem is to find $\Delta F(x)$, $\Delta G(x)$, $A(x)$ and $B(x)$ satisfying Eq. (4) while making $\|\Delta F\|_2^2 + \|\Delta G\|_2^2$ as small as possible.

2.1. The Real Coefficient Case

Assuming that we have $F(x)$ and $G(x)$ as polynomials with the real coefficients and find an approximate GCD with the real coefficients as well, we represent $\tilde{F}(x)$, $\tilde{G}(x)$, $A(x)$ and $B(x)$ with the real coefficients as

$$\begin{aligned} \tilde{F}(x) &= \tilde{f}_m x^m + \dots + \tilde{f}_0 x^0, & \tilde{G}(x) &= \tilde{g}_n x^n + \dots + \tilde{g}_0 x^0, \\ A(x) &= a_{n-d} x^{n-d} + \dots + a_0 x^0, & B(x) &= b_{m-d} x^{m-d} + \dots + b_0 x^0, \end{aligned} \quad (5)$$

respectively, thus $\|\Delta F\|_2^2 + \|\Delta G\|_2^2$ and Eq. (4) become as

$$\|\Delta F\|_2^2 + \|\Delta G\|_2^2 = (\tilde{f}_m - f_m)^2 + \cdots + (\tilde{f}_0 - f_0)^2 + (\tilde{g}_n - g_n)^2 + \cdots + (\tilde{g}_0 - g_0)^2, \quad (6)$$

$$N_{d-1}(\tilde{F}, \tilde{G}) \cdot \mathbf{v} = \mathbf{0}, \quad (7)$$

respectively, with $N_j(\tilde{F}, \tilde{G})$ as in (3) and

$$\mathbf{v} = {}^t(a_{n-d}, \dots, a_0, b_{m-d}, \dots, b_0). \quad (8)$$

Then, Eq. (7) is regarded as a system of $m + n - d + 1$ equations in $\tilde{f}_m, \dots, \tilde{f}_0, \tilde{g}_n, \dots, \tilde{g}_0, a_{n-d}, \dots, a_0, b_{m-d}, \dots, b_0$, as

$$q_1 = \tilde{f}_m a_{n-d} + \tilde{g}_n b_{m-d} = 0, \dots, q_{m+n-d+1} = \tilde{f}_0 a_0 + \tilde{g}_0 b_0 = 0, \quad (9)$$

by putting q_j as the j -th row. Furthermore, for solving the problem below stably, we add another constraint enforcing the coefficients of $A(x)$ and $B(x)$ such that $\|A(x)\|_2^2 + \|B(x)\|_2^2 = 1$; thus we add

$$q_0 = a_{n-d}^2 + \cdots + a_0^2 + b_{m-d}^2 + \cdots + b_0^2 - 1 = 0 \quad (10)$$

into Eq. (9).

Now, we substitute the variables

$$(\tilde{f}_m, \dots, \tilde{f}_0, \tilde{g}_n, \dots, \tilde{g}_0, a_{n-d}, \dots, a_0, b_{m-d}, \dots, b_0) \quad (11)$$

as $\mathbf{x} = (x_1, \dots, x_{2(m+n-d+2)})$, thus Eq. (6) and (9) with (10) become

$$f(\mathbf{x}) = (x_1 - f_m)^2 + \cdots + (x_{m+1} - f_0)^2 + (x_{m+2} - g_n)^2 + \cdots + (x_{m+n+2} - g_0)^2, \quad (12)$$

$$\mathbf{q}(\mathbf{x}) = {}^t(q_0(\mathbf{x}), q_1(\mathbf{x}), \dots, q_{m+n-d+1}(\mathbf{x})) = \mathbf{0}, \quad (13)$$

respectively. Therefore, the problem of finding an approximate GCD can be formulated as a constrained minimization problem of finding a minimizer of the objective function $f(\mathbf{x})$ in (12), subject to $\mathbf{q}(\mathbf{x}) = \mathbf{0}$ in Eq. (13).

2.2. The Complex Coefficient Case

Now let us assume that we have $F(x)$ and $G(x)$ with the complex coefficients in general, represented as

$$\begin{aligned} F(x) &= (f_{m,1} + f_{m,2}\mathbf{i})x^m + \cdots + (f_{0,1} + f_{0,2}\mathbf{i}), \\ G(x) &= (g_{n,1} + g_{n,2}\mathbf{i})x^n + \cdots + (g_{0,1} + g_{0,2}\mathbf{i}), \end{aligned}$$

where $f_{j,1}, g_{j,1}, f_{j,2}, g_{j,2}$ are real numbers; $f_{j,1}$, and $g_{j,1}$ represent the real parts; $f_{j,2}, g_{j,2}$ represent the imaginary parts, with \mathbf{i} as the imaginary unit, and find

an approximate GCD with the complex coefficients. Then, we represent $\tilde{F}(x)$, $\tilde{G}(x)$, $A(x)$ and $B(x)$ with the complex coefficients as

$$\begin{aligned}\tilde{F}(x) &= (\tilde{f}_{m,1} + \tilde{f}_{m,2}\mathbf{i})x^m + \cdots + (\tilde{f}_{0,1} + \tilde{f}_{0,2}\mathbf{i})x^0, \\ \tilde{G}(x) &= (\tilde{g}_{n,1} + \tilde{g}_{n,2}\mathbf{i})x^n + \cdots + (\tilde{g}_0x^0 + \tilde{g}_{0,2}\mathbf{i})x^0, \\ A(x) &= (a_{n-d,1} + a_{n-d,2}\mathbf{i})x^{n-d} + \cdots + (a_{0,1} + a_{0,2}\mathbf{i})x^0, \\ B(x) &= (b_{m-d,1} + b_{m-d,2}\mathbf{i})x^{m-d} + \cdots + (b_{0,1} + b_{0,2}\mathbf{i})x^0,\end{aligned}\tag{14}$$

respectively, where $\tilde{f}_{j,1}$, $\tilde{f}_{j,2}$, $\tilde{g}_{j,1}$, $\tilde{g}_{j,2}$, $a_{j,1}$, $a_{j,2}$, $b_{j,1}$, $b_{j,2}$ are real numbers.

For the objective function, $\|\Delta F\|_2^2 + \|\Delta G\|_2^2$ becomes as

$$\sum_{j=0}^m [(\tilde{f}_{j,1} - f_{j,1})^2 + (\tilde{f}_{j,2} - f_{j,2})^2] + \sum_{j=0}^n [(\tilde{g}_{j,1} - g_{j,1})^2 + (\tilde{g}_{j,2} - g_{j,2})^2]. \tag{15}$$

For the constraint, Eq. (4) becomes as

$$\begin{pmatrix} \tilde{f}_{m,1} + \tilde{f}_{m,2}\mathbf{i} & & \tilde{g}_{n,1} + \tilde{g}_{n,2}\mathbf{i} & \\ & \ddots & & \\ \tilde{f}_{0,1} + \tilde{f}_{0,2}\mathbf{i} & \tilde{f}_{m,1} + \tilde{f}_{m,2}\mathbf{i} & \tilde{g}_{0,1} + \tilde{g}_{0,2}\mathbf{i} & \tilde{g}_{n,1} + \tilde{g}_{n,2}\mathbf{i} \\ & & & \\ & & \ddots & \\ & \tilde{f}_{0,1} + \tilde{f}_{0,2}\mathbf{i} & & \tilde{g}_{0,1} + \tilde{g}_{0,2}\mathbf{i} \end{pmatrix} \times \begin{pmatrix} a_{n-d,1} + a_{n-d,2}\mathbf{i} \\ \vdots \\ a_{0,1} + a_{0,2}\mathbf{i} \\ b_{m-d,1} + b_{m-d,2}\mathbf{i} \\ \vdots \\ b_{0,1} + b_{0,2}\mathbf{i} \end{pmatrix} = \mathbf{0}. \tag{16}$$

By expressing the subresultant matrix and the column vector in (16) separated into the real and the complex parts, respectively, we express (16) as

$$(N_1 + N_2\mathbf{i})(\mathbf{v}_1 + \mathbf{v}_2\mathbf{i}) = \mathbf{0}, \tag{17}$$

with

$$\begin{aligned}N_1 &= \begin{pmatrix} \tilde{f}_{m,1} & & \tilde{g}_{n,1} & \\ \vdots & \ddots & & \\ \tilde{f}_{0,1} & \tilde{f}_{m,1} & \tilde{g}_{0,1} & \tilde{g}_{n,1} \\ & & & \\ & & \ddots & \\ & \tilde{f}_{0,1} & & \tilde{g}_{0,1} \end{pmatrix}, \quad N_2 = \begin{pmatrix} \tilde{f}_{m,2} & & \tilde{g}_{n,2} & \\ \vdots & \ddots & & \\ \tilde{f}_{0,2} & \tilde{f}_{m,2} & \tilde{g}_{0,2} & \tilde{g}_{n,2} \\ & & & \\ & & \ddots & \\ & \tilde{f}_{0,2} & & \tilde{g}_{0,2} \end{pmatrix}, \\ \mathbf{v}_1 &= {}^t(a_{n-d,1}, \dots, a_{0,1}, b_{m-d,1}, \dots, b_{0,1}), \\ \mathbf{v}_2 &= {}^t(a_{n-d,2}, \dots, a_{0,2}, b_{m-d,2}, \dots, b_{0,2}).\end{aligned}\tag{18}$$

We can expand the left-hand-side of Eq. (17) as

$$(N_1 + N_2 \mathbf{i})(\mathbf{v}_1 + \mathbf{v}_2 \mathbf{i}) = (N_1 \mathbf{v}_1 - N_2 \mathbf{v}_2) + \mathbf{i}(N_1 \mathbf{v}_2 + N_2 \mathbf{v}_1),$$

thus, Eq. (17) is equivalent to a system of equations

$$N_1 \mathbf{v}_1 - N_2 \mathbf{v}_2 = \mathbf{0}, \quad N_1 \mathbf{v}_2 + N_2 \mathbf{v}_1 = \mathbf{0},$$

which is expressed as

$$\begin{pmatrix} N_1 & -N_2 \\ N_2 & N_1 \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \end{pmatrix} = \mathbf{0}. \quad (19)$$

Furthermore, as well as in the real coefficients case, we add another constraint for the coefficient of $A(x)$ and $B(x)$ as

$$\begin{aligned} \|A(x)\|_2^2 + \|B(x)\|_2^2 &= (a_{n-d,1}^2 + \cdots + a_{0,1}^2) + (b_{m-d,1}^2 + \cdots + b_{0,1}^2) \\ &+ (a_{n-d,2}^2 + \cdots + a_{0,2}^2) + (b_{m-d,2}^2 + \cdots + b_{0,2}^2) - 1 = 0, \end{aligned} \quad (20)$$

which can be expressed together with (19) as

$$\begin{pmatrix} {}^t\mathbf{v}_1 & {}^t\mathbf{v}_2 & -1 \\ N_1 & -N_2 & \mathbf{0} \\ N_2 & N_1 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ 1 \end{pmatrix} = \mathbf{0}, \quad (21)$$

where Eq. (20) has been put on the top of Eq. (19). Note that, in Eq. (21), we have total of $2(m+n-d+1)+1$ equations in the coefficients of polynomials in (14) as a constraint, with the j -th row of which is expressed as $q_j = 0$, as similarly as in the real case (9) with (10).

Now, as in the real case, we substitute the variables

$$\begin{aligned} &(\tilde{f}_{m,1}, \dots, \tilde{f}_{0,1}, \tilde{g}_{n,1}, \dots, \tilde{g}_{0,1}, \tilde{f}_{m,2}, \dots, \tilde{f}_{0,2}, \tilde{g}_{n,2}, \dots, \tilde{g}_{0,2}, \\ &a_{n-d,1}, \dots, a_{0,1}, b_{m-d,1}, \dots, b_{0,1}, a_{n-d,2}, \dots, a_{0,2}, b_{m-d,2}, \dots, b_{0,2}) \end{aligned} \quad (22)$$

as $\mathbf{x} = (x_1, \dots, x_{4(m+n-d+2)})$, thus Eq. (15) and (21) become as

$$\begin{aligned} f(\mathbf{x}) &= (x_1 - f_{m,1})^2 + \cdots + (x_{m+1} - f_{0,1})^2 \\ &+ (x_{m+2} - g_{n,1})^2 + \cdots + (x_{m+n+2} - g_{0,1})^2 \\ &+ (x_{m+n+3} - f_{m,2})^2 + \cdots + (x_{2m+n+3} - f_{0,2})^2 \\ &+ (x_{2m+n+4} - g_{n,2})^2 + \cdots + (x_{2(m+n+2)} - g_{0,2})^2, \end{aligned} \quad (23)$$

$$\mathbf{q}(\mathbf{x}) = {}^t(q_1(\mathbf{x}), \dots, q_{2(m+n-d+1)+1}(\mathbf{x})) = \mathbf{0}, \quad (24)$$

respectively. Therefore, the problem of finding an approximate GCD can be formulated as a constrained minimization problem of finding a minimizer of the objective function $f(\mathbf{x})$ in Eq. (23), subject to $\mathbf{q}(\mathbf{x}) = \mathbf{0}$ in Eq. (24).

3. The Gradient-Projection Method and the Modified Newton Method

In this section, we consider the problem of minimizing an objective function $f(\mathbf{x}) : \mathbf{R}^n \rightarrow \mathbf{R}$, subject to the constraints $\mathbf{q}(\mathbf{x}) = \mathbf{0}$ for $\mathbf{q}(\mathbf{x}) = {}^t(q_1(\mathbf{x}), q_2(\mathbf{x}), \dots, q_m(\mathbf{x}))$, with $m \leq n$, where $q_j(\mathbf{x})$ is a function of $\mathbf{R}^n \rightarrow \mathbf{R}$, and $f(\mathbf{x})$ and $q_j(\mathbf{x})$ are twice continuously differentiable (here, we refer presentations of the problem to Tanabe [24] and the references therein).

If we assume that the Jacobian matrix

$$J_{\mathbf{q}}(\mathbf{x}) = \left(\frac{\partial q_i}{\partial x_j} \right)$$

is of full rank, or

$$\text{rank}(J_{\mathbf{q}}(\mathbf{x})) = m, \quad (25)$$

on the feasible region $V_{\mathbf{q}}$ defined by

$$V_{\mathbf{q}} = \{\mathbf{x} \in \mathbf{R}^n \mid \mathbf{q}(\mathbf{x}) = \mathbf{0}\},$$

then the feasible region $V_{\mathbf{q}}$ is an $(n - m)$ -dimensional differential manifold in \mathbf{R}^n and f is differentiable function on the manifold $V_{\mathbf{q}}$. Thus, our problem is to find a point in $V_{\mathbf{q}}$, which will be a candidate of a local minimizer, satisfying the well-known “first-order necessary conditions” (for the proof, refer to the literature on optimization such as Nocedal and Wright [16]).

Theorem 1 (First-order necessary conditions). *Suppose that $\mathbf{x}^* \in V_{\mathbf{q}}$ is a local solution of the problem in the above, that the functions $f(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ are continuously differentiable at \mathbf{x}^* , and that we have (25) at \mathbf{x}^* . Then, there exist a Lagrange multiplier vector $\boldsymbol{\lambda}^* \in \mathbf{R}^m$ satisfying*

$$\nabla f(\mathbf{x}^*) - {}^t(J_{\mathbf{q}}(\mathbf{x}^*))\boldsymbol{\lambda}^* = \mathbf{0}, \quad \mathbf{q}(\mathbf{x}^*) = \mathbf{0}. \quad \square$$

3.1. The Gradient-Projection Method

Let $\mathbf{x}_k \in \mathbf{R}^n$ be a feasible point, or a point satisfying $\mathbf{x}_k \in V_{\mathbf{q}}$. Rosen’s gradient projection method ([20]) is based on projecting the steepest descent direction onto the tangent space of the manifold $V_{\mathbf{q}}$ at \mathbf{x}_k , which is denoted to $T_{\mathbf{x}_k}$ and represented by the kernel of the Jacobian matrix $J_{\mathbf{q}}(\mathbf{x}_k)$ as

$$T_{\mathbf{x}_k} = \ker(J_{\mathbf{q}}(\mathbf{x}_k)) = \{\mathbf{z} \in \mathbf{R}^n \mid J_{\mathbf{q}}(\mathbf{x}_k)\mathbf{z} = \mathbf{0} \in \mathbf{R}^m\}. \quad (26)$$

We have steepest descent direction of the objective function f at \mathbf{x}_k as

$$-\nabla f(\mathbf{x}_k) = -{}^t \left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n} \right). \quad (27)$$

Then, the search direction \mathbf{d}_k is defined by the projection of the steepest descent direction of f in (27) onto $T_{\mathbf{x}_k}$ in (26) as

$$\mathbf{d}_k = -P(\mathbf{x}_k)\nabla f(\mathbf{x}_k). \quad (28)$$

Here, $P(\mathbf{x}_k)$ is the orthogonal projection operator on $T_{\mathbf{x}_k}$ defined as

$$P(\mathbf{x}_k) = I - (J_{\mathbf{q}}(\mathbf{x}_k))^+(J_{\mathbf{q}}(\mathbf{x}_k)),$$

where I is the identity matrix and $(J_{\mathbf{q}}(\mathbf{x}_k))^+$ is the Moore-Penrose inverse of $(J_{\mathbf{q}}(\mathbf{x}_k))$. Under the assumption (25), we have

$$(J_{\mathbf{q}}(\mathbf{x}_k))^+ = {}^t(J_{\mathbf{q}}(\mathbf{x}_k)) \cdot (J_{\mathbf{q}}(\mathbf{x}_k) \cdot {}^t(J_{\mathbf{q}}(\mathbf{x}_k)))^{-1}$$

(see Tanabe [24, Eq. (8)]).

With an appropriate step width α_k (see Remark 1) satisfying $0 < \alpha_k \leq 1$, let

$$\mathbf{y}_k = \mathbf{x}_k + \alpha_k \cdot \mathbf{d}_k.$$

Since $V_{\mathbf{q}}$ is nonlinear in general, \mathbf{y}_k may not in $V_{\mathbf{q}}$: in such a case, we take a *restoration* move to bring \mathbf{y}_k back to $V_{\mathbf{q}}$, as follows. Let $\mathbf{x} \in \mathbf{R}^n$ be an arbitrary point. Then, at \mathbf{y}_k , the constraint $\mathbf{q}(\mathbf{x})$ can be linearly approximated as

$$\mathbf{q}(\mathbf{y}_k + \mathbf{x}) \simeq \mathbf{q}(\mathbf{y}_k) + J_{\mathbf{q}}(\mathbf{y}_k)\mathbf{x}.$$

Assuming $\mathbf{y}_k + \mathbf{x} \in V_{\mathbf{q}}$, we have $\mathbf{q}(\mathbf{y}_k + \mathbf{x}) = \mathbf{0}$ thus the approximation of \mathbf{x} can be calculated as

$$\mathbf{x} = -(J_{\mathbf{q}}(\mathbf{y}_k))^+ \mathbf{q}(\mathbf{y}_k). \quad (29)$$

If \mathbf{y}_k is sufficiently close to $V_{\mathbf{q}}$, then we can restore \mathbf{y}_k back onto $V_{\mathbf{q}}$ by applying (29) iteratively for several times. Note that the restoration move can also be used in the case the initial point of the minimization process is away from the feasible region $V_{\mathbf{q}}$.

Summarizing the above, we obtain an algorithm for the gradient projection as follows.

Algorithm 1 (The gradient-projection method ([20])).

Step 1 [Restoration] If the given point \mathbf{x}_0 does not satisfy $\mathbf{x}_0 \in V_{\mathbf{q}}$, first move \mathbf{x}_0 onto $V_{\mathbf{q}}$ by the iteration of Eq. (29), then let \mathbf{x}_0 be the restored point on $V_{\mathbf{q}}$. Let $k = 0$.

Step 2 [Projection] For \mathbf{x}_k , calculate $\mathbf{d}_k = -P(\mathbf{x}_k)\nabla f(\mathbf{x}_k)$ by (28). If $\|\mathbf{d}_k\|$ is sufficiently small for an appropriate norm, go to Step 4. Otherwise, calculate the step width α_k by an appropriate line search method (see Remark 1) then let $\mathbf{y}_{k,0} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$.

Step 3 [Restoration] If $\mathbf{q}(\mathbf{y}_{k,0}) \neq \mathbf{0}$, move $\mathbf{y}_{k,0}$ back onto $V_{\mathbf{q}}$ iteratively by (29). Let $\mathbf{y}_{k,l+1} = \mathbf{y}_{k,l} - (J_{\mathbf{q}}(\mathbf{y}_{k,l}))^+ \mathbf{q}(\mathbf{y}_{k,l})$ for $l = 0, 1, 2, \dots$. When $\mathbf{y}_{k,l}$ satisfies $\mathbf{q}(\mathbf{y}_{k,l}) \simeq \mathbf{0}$, then let $\mathbf{x}_{k+1} = \mathbf{y}_{k,l}$ and go to Step 2.

Step 4 [Checking the first-order necessary conditions] If \mathbf{x}_k satisfies Theorem 1, then return \mathbf{x}_k .

Remark 1. Choosing appropriate step width in the iteration is a fundamental issue in optimization method and is discussed in standard literature of optimization (e.g. [16]). Although we simply set $\alpha_k = 1$ in our implementation, more sophisticated calculation of step width might improve accuracy and/or convergence of the algorithm (see also concluding remarks (Section 6)).

3.2. The Modified Newton Method

The modified Newton method by Tanabe [24] is a generalization of the Newton's method, which derives several different methods, by modifying the Hessian of the Lagrange function. A generalization of the gradient-projection method combines the *restoration step* and the *projection step* in Algorithm 1. For $\mathbf{x}_k \in V_{\mathbf{q}}$, we calculate the search direction \mathbf{d}_k , along with the associated Lagrange multipliers $\boldsymbol{\lambda}_{k+1}$, by solving a linear system

$$\begin{pmatrix} I & -{}^t(J_{\mathbf{q}}(\mathbf{x}_k)) \\ J_{\mathbf{q}}(\mathbf{x}_k) & \mathbf{O} \end{pmatrix} \begin{pmatrix} \mathbf{d}_k \\ \boldsymbol{\lambda}_{k+1} \end{pmatrix} = - \begin{pmatrix} \nabla f(\mathbf{x}_k) \\ \mathbf{q}(\mathbf{x}_k) \end{pmatrix}, \quad (30)$$

then put $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \cdot \mathbf{d}_k$ with an appropriate step width α_k . Solving Eq. (30) under assumption (25), we have

$$\begin{aligned} \mathbf{d}_k &= -P(\mathbf{x}_k)\nabla f(\mathbf{x}_k) - (J_{\mathbf{q}}(\mathbf{x}_k))^+ \mathbf{q}(\mathbf{x}_k), \\ \boldsymbol{\lambda}_{k+1} &= {}^t((J_{\mathbf{q}}(\mathbf{x}_k))^+ \nabla f(\mathbf{x}_k) - (J_{\mathbf{q}}(\mathbf{x}_k) \cdot {}^t(J_{\mathbf{q}}(\mathbf{x}_k)))^{-1} \mathbf{q}(\mathbf{x}_k)). \end{aligned} \quad (31)$$

Note that, in \mathbf{d}_k in (31), the term $-P(\mathbf{x}_k)\nabla f(\mathbf{x}_k)$ comes from the projection (28), while another term $-(J_{\mathbf{q}}(\mathbf{x}_k))^+ \mathbf{q}(\mathbf{x}_k)$ comes from the restoration (29). If we have $\mathbf{x}_k \in V_{\mathbf{q}}$, the iteration formula (30) is equivalent to the projection (28). After an iteration, the new estimate \mathbf{x}_{k+1} may not satisfy $\mathbf{x}_{k+1} \in V_{\mathbf{q}}$: in such a case, in the next iteration, the point will be pulled back onto $V_{\mathbf{q}}$ by the $-(J_{\mathbf{q}}(\mathbf{x}_k))^+ \mathbf{q}(\mathbf{x}_k)$ term. Therefore, by solving Eq. (30) iteratively, we expect that the approximations \mathbf{x}_k moves toward descending direction of f along with tracing the feasible set $V_{\mathbf{q}}$.

Summarizing the above, we obtain an algorithm as follows.

Algorithm 2 (The modified Newton method ([24])).

Step 1 [Finding a search direction] For \mathbf{x}_k , calculate \mathbf{d}_k by solving the linear system (30). If $\|\mathbf{d}_k\|$ is sufficiently small, go to Step 2. Otherwise, calculate the step width α_k by an appropriate line search method (see Remark 1), let $\mathbf{x}_{k+1} = \mathbf{x}_k + \alpha_k \mathbf{d}_k$, then go to Step 1.

Step 2 [Checking the first-order necessary conditions] If \mathbf{x}_k satisfies Theorem 1 with sufficient accuracy, then return \mathbf{x}_k .

4. The Algorithm for Approximate GCD

In applying the gradient-projection method or the modified Newton method to the approximate GCD problem, we discuss issues in the construction of the algorithm in detail, such as

- Representation of the Jacobian matrix $J_{\mathbf{q}}(\mathbf{x})$ (Section 4.1),
- Stability of the algorithm by certifying that $J_{\mathbf{q}}(\mathbf{x})$ has full rank (Section 4.2),

- Setting the initial values (Section 4.3),
- Regarding the minimization problem as the minimum distance problem (Section 4.4),
- Calculating the actual GCD and correcting the coefficients of \tilde{F} and \tilde{G} (Section 4.5),

as follows. After presenting the algorithm, we give a modification for preserving monicity for the real coefficient case and running time analysis, and end this section with examples.

4.1. Representation of the Jacobian Matrix

For a polynomial $P(x) \in \mathbf{R}[x]$ or $\mathbf{C}[x]$ represented as

$$P(x) = p_n x^n + \cdots + p_0 x^0,$$

let $C_k(P)$ be a complex $(n+k, k+1)$ matrix defined as

$$C_k(P) = \underbrace{\begin{pmatrix} p_n & & & \\ \vdots & \ddots & & \\ p_0 & & p_n & \\ & \ddots & \vdots & \\ & & & p_0 \end{pmatrix}}_{k+1}.$$

We show the Jacobian matrix in the real and the complex coefficient cases, both of which can easily be constructed in every iteration in Algorithms 1 and 2.

4.1.1. The Real Coefficient Case

For co-factors $A(x)$ and $B(x)$ as in (5), consider matrices $C_m(A)$ and $C_n(B)$. Then, by the definition of the constraint (13), we have the Jacobian matrix $J_{\mathbf{q}}(\mathbf{x})$ (with the original notation of variables for \mathbf{x} as in (11)) as

$$J_{\mathbf{q}}(\mathbf{x}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & 2 \cdot {}^t \mathbf{v} \\ C_m(A) & C_n(B) & N_{d-1}(\tilde{F}, \tilde{G}) \end{pmatrix}, \quad (32)$$

with $N_j(\tilde{F}, \tilde{G})$ as in (3) and \mathbf{v} as in (8), respectively. Note that the matrix $J_{\mathbf{q}}(\mathbf{x})$ has $m+n-d+2$ rows and $2(m+n-d+2)$ columns.

4.1.2. The Complex Coefficient Case

For co-factors $A(x)$ and $B(x)$ as in (14), consider matrices $C_m(A)$ and $C_n(B)$ and express them as the sum of matrices consisting of the real and the imaginary

parts of whose elements, respectively, as

$$\begin{aligned}
C_m(A) &= \begin{pmatrix} a_{n-d,1} & & & \\ \vdots & \ddots & & \\ a_{0,1} & & a_{n-d,1} & \\ & \ddots & \vdots & \\ & & a_{0,1} & \end{pmatrix} + i \begin{pmatrix} a_{n-d,2} & & & \\ \vdots & \ddots & & \\ a_{0,2} & & a_{n-d,2} & \\ & \ddots & \vdots & \\ & & a_{0,2} & \end{pmatrix} \\
&= C_m(A)_1 + i C_m(A)_2, \\
C_n(B) &= \begin{pmatrix} b_{m-d,1} & & & \\ \vdots & \ddots & & \\ b_{0,1} & & b_{m-d,1} & \\ & \ddots & \vdots & \\ & & b_{0,1} & \end{pmatrix} + i \begin{pmatrix} b_{m-d,2} & & & \\ \vdots & \ddots & & \\ b_{0,2} & & b_{m-d,2} & \\ & \ddots & \vdots & \\ & & b_{0,2} & \end{pmatrix} \\
&= C_n(B)_1 + i C_n(B)_2,
\end{aligned}$$

respectively, and define

$$\begin{aligned}
A_1 &= [C_m(A)_1 \ C_n(B)_1] = \begin{pmatrix} a_{n-d,1} & & b_{m-d,1} & \\ \vdots & \ddots & \vdots & \ddots \\ a_{0,1} & a_{n-d,1} & b_{0,1} & b_{m-d,1} \\ & \ddots & \vdots & \ddots \\ & & a_{0,1} & b_{0,1} \end{pmatrix}, \\
A_2 &= [C_m(A)_2 \ C_n(B)_2] = \begin{pmatrix} a_{n-d,2} & & b_{m-d,2} & \\ \vdots & \ddots & \vdots & \ddots \\ a_{0,2} & a_{n-d,2} & b_{0,2} & b_{m-d,2} \\ & \ddots & \vdots & \ddots \\ & & a_{0,2} & b_{0,2} \end{pmatrix}.
\end{aligned} \tag{33}$$

(Note that A_1 and A_2 are matrices of the real numbers of $m + n - d + 1$ rows and $m + n + 2$ columns.) Then, by the definition of the constraint (24), we have the Jacobian matrix $J_q(\mathbf{x})$ (with the original notation of variables for \mathbf{x} as in (22)) as

$$J_q(\mathbf{x}) = \begin{pmatrix} \mathbf{0} & \mathbf{0} & 2 \cdot {}^t \mathbf{v}_1 & 2 \cdot {}^t \mathbf{v}_2 \\ A_1 & -A_2 & N_1 & -N_2 \\ A_2 & A_1 & N_2 & N_1 \end{pmatrix}, \tag{34}$$

with A_1 and A_2 as in (33) and N_1 , N_2 , \mathbf{v}_1 and \mathbf{v}_2 as in (18), respectively.

4.2. Stability of the Algorithm

In this paper, we treat the notion of “stability” of the algorithm as to keep that the Jacobian $J_q(\mathbf{x})$ in Algorithms 1 and 2 has full rank, whereas we usually discuss *stability* as a notion in backward and/or forward error analysis of numerical algorithms [11].

In executing Algorithm 1 or 2, we need the algorithm to be *stable* in the sense that we need to keep that $J_{\mathbf{q}}(\mathbf{x})$ has full rank: otherwise, we cannot correctly calculate $(J_{\mathbf{q}}(\mathbf{x}))^+$ (in Algorithm 1) or the matrix in (30) becomes singular (in Algorithm 2) thus we are unable to decide proper search direction. For this requirement, we have the following observations.

Proposition 1. *Let $\mathbf{x}^* \in V_{\mathbf{q}}$ be any feasible point satisfying Eq. (13). Then, if the corresponding polynomials do not have a GCD whose degree exceeds d , then $J_{\mathbf{q}}(\mathbf{x}^*)$ has full rank.*

Proof. We prove the proposition in the real and the complex coefficient cases separately.

4.2.1. The Real Coefficient Case

Let $\mathbf{x}^* = (\tilde{f}_m, \dots, \tilde{f}_0, \tilde{g}_n, \dots, \tilde{g}_0, a_{n-d}, \dots, a_0, b_{m-d}, \dots, b_0)$ with its polynomial representation expressed as in (5) (note that this assumption permits the polynomials $\tilde{F}(x)$ and $\tilde{G}(x)$ to be relatively prime in general). To verify our claim, we show that we have $\text{rank}(J_{\mathbf{q}}(\mathbf{x}^*)) = m + n - d + 2$ with $J_{\mathbf{q}}(\mathbf{x}^*)$ as in (32). Let us express $J_{\mathbf{q}}(\mathbf{x}^*) = (J_L \mid J_R)$, where J_L and J_R are column blocks expressed as

$$J_L = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ C_m(A) & C_n(B) \end{pmatrix}, \quad J_R = \begin{pmatrix} 2 \cdot \mathbf{v} \\ N_{d-1}(\tilde{F}, \tilde{G}) \end{pmatrix},$$

respectively. Then, we have the following lemma.

Lemma 1. *We have $\text{rank}(J_L) = m + n - d + 1$.*

Proof. Let us express $J_L = (J_{LL} \mid J_{LR})$, where

$$J_{LL} = \begin{pmatrix} \mathbf{0} \\ C_m(A) \end{pmatrix}, \quad J_{LR} = \begin{pmatrix} \mathbf{0} \\ C_n(B) \end{pmatrix},$$

and let \bar{J}_L be a submatrix of J_L by taking the right $m - d$ columns of J_{LL} and the right $n - d$ columns of J_{LR} . Then, we see that the bottom $m + n - 2d$ rows of \bar{J}_L is equal to $N(A, B)$, the Sylvester matrix of $A(x)$ and $B(x)$. By the assumption, polynomials $A(x)$ and $B(x)$ are relatively prime, and there exist no nonzero elements in \bar{J}_L except for the bottom $m + n - 2d$ rows, we have $\text{rank}(\bar{J}_L) = m + n - 2d$.

By the above structure of \bar{J}_L and the lower triangular structure of J_{LL} and J_{LR} , we can take the left $d + 1$ columns of J_{LL} or J_{LR} satisfying linear independence along with the $m + n - 2d$ columns in \bar{J}_L . Therefore, these $m + n - d + 1$ columns generate a $(m + n - d + 1)$ -dimensional subspace in $\mathbf{R}^{m+n-d+2}$ satisfying

$$\{^t(x_1, \dots, x_{m+n-d+2}) \in \mathbf{R}^{m+n-d+2} \mid x_1 = 0\}, \quad (35)$$

and we see that none of the columns in J_L have nonzero element in the top coordinate. This proves the lemma. \square

Proof of Proposition 1 (in the real coefficient case, continued). By the assumptions, we have at least one column vector in J_R with nonzero coordinate on the top row. By adding such a column vector to the basis of the subspace (35) that are generated as in Lemma 1, we have a basis of $\mathbf{R}^{m+n-d+2}$. This implies $\text{rank}(J_{\mathbf{q}}(\mathbf{x})) = m + n - d + 2$, which proves the proposition in the real coefficient case.

4.2.2. The Complex Coefficient Case

Let $\mathbf{x}^* = (\tilde{f}_{m,1}, \dots, \tilde{f}_{0,1}, \tilde{g}_{n,1}, \dots, \tilde{g}_{0,1}, \tilde{f}_{m,2}, \dots, \tilde{f}_{0,2}, \tilde{g}_{n,2}, \dots, \tilde{g}_{0,2}, a_{n-d,1}, \dots, a_{0,1}, b_{m-d,1}, \dots, b_{0,1}, a_{n-d,2}, \dots, a_{0,2}, b_{m-d,2}, \dots, b_{0,2})$ with its polynomial representation expressed as in (14) (note that this assumption permits the polynomials $\tilde{F}(x)$ and $\tilde{G}(x)$ to be relatively prime in general). To verify our claim, we show that we have $\text{rank}(J_{\mathbf{q}}(\mathbf{x}^*)) = 2(m + n - d + 1) + 1$ as in (25), with $J_{\mathbf{q}}(\mathbf{x}^*)$ as in (34). Let us express $J_{\mathbf{q}}(\mathbf{x}^*) = (J_L \mid J_R)$, where J_L and J_R are column blocks expressed as

$$J_L = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix}, \quad J_R = \begin{pmatrix} 2 \cdot {}^t \mathbf{v}_1 & 2 \cdot {}^t \mathbf{v}_2 \\ N_1 & -N_2 \\ N_2 & N_1 \end{pmatrix},$$

respectively. Then, we have the following lemma.

Lemma 2. *We have $\text{rank}(J_L) = 2(m + n - d + 1)$.*

Proof. For $A_1 = [C_m(A)_1 \ C_n(B)_1]$, let $\overline{C_m(A)_1}$ be the right $m - d$ columns of $C_m(A)_1$ and $\overline{C_n(B)_1}$ be the right $n - d$ columns of $C_n(B)_1$. Then, we see that the bottom $m + n - 2d$ rows of the matrix $\bar{C} = [\overline{C_m(A)_1} \ \overline{C_n(B)_1}]$ is equal to the matrix consisting of the real part of the elements of $N(A, B)$, the Sylvester matrix of $A(x)$ and $B(x)$. By the assumption, polynomials $A(x)$ and $B(x)$ are relatively prime, and there exist no nonzero elements in \bar{C} except for the bottom $m + n - 2d$ rows, thus we have $\text{rank}(\bar{C}) = m + n - 2d$.

By the structure of \bar{C} and the lower triangular structure of $C_m(A)_1$ and $C_n(B)_1$, we can take the left $d + 1$ columns of $C_m(A)_1$ or $C_n(B)_1$ satisfying linear independence along with \bar{C} , which implies that there exist a nonsingular square matrix T of order $m + n + 2$ satisfying

$$A_1 T = R, \tag{36}$$

where R is a lower triangular matrix, thus we have $\text{rank}(A_1) = \text{rank}(R) = m + n - d + 1$.

Furthermore, by using T and R in (36), we have

$$\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ A_1 & -A_2 \\ A_2 & A_1 \end{pmatrix} \begin{pmatrix} T & \mathbf{0} \\ \mathbf{0} & T \end{pmatrix} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ R & -A_2 T \\ A_2 T & R \end{pmatrix}, \tag{37}$$

followed by a suitable transformation on columns on the matrix in the right-hand-side of (37), we can make A_2T to zero matrix, which implies that

$$\text{rank}(J_L) = \text{rank} \left(\begin{pmatrix} \mathbf{0} & \mathbf{0} \\ R & -A_2T \\ A_2T & R \end{pmatrix} \right) = 2 \cdot \text{rank}(R) = 2(m + n - d + 1).$$

This proves the lemma. \square

Proof of Proposition 1 (in the complex coefficient case, continued). By the assumptions, we have at least one nonzero coordinate in the top row in J_R , while we have no nonzero coordinate in the top row in J_L , thus we have $\text{rank}(J_q(\mathbf{x})) = 2(m + n - d + 1) + 1$, which proves the proposition in the complex coefficient case. \square

Remark 2. Proposition 1 says that, so long as the search direction in the minimization problem satisfies that corresponding polynomials have a GCD of degree not exceeding d , then $J_q(\mathbf{x})$ has full rank, thus we can safely calculate the next search direction for approximate GCD. On the other hand, it is still not clear when $J_q(\mathbf{x})$ becomes singular in our minimization problem. Although our experiments have shown that the iteration converges for any d satisfying $0 < d \leq n$ in many examples, its theoretical property deserves further investigation (see also concluding remarks (Section 6)).

4.3. Setting the Initial Values

At the beginning of iterations, we give the initial value \mathbf{x}_0 by using the singular value decomposition (SVD) ([9]), as follows.

4.3.1. The Real Coefficient Case

In the case of the real coefficients, we calculate the SVD of the $(d - 1)$ -th subresultant matrix $N_{d-1}(F, G) : \mathbf{R}^{m+n-2d+2} \rightarrow \mathbf{R}^{m+n-d+1}$ (see (3)). Let $N_{d-1}(F, G) = U \Sigma^t V$ be the SVD of $N_{d-1}(F, G)$, where

$$\begin{aligned} N_{d-1}(F, G) &= U \Sigma^t V, \quad U = (\mathbf{u}_1, \dots, \mathbf{u}_{m+n-2d+2}), \\ \Sigma &= \text{diag}(\sigma_1, \dots, \sigma_{m+n-2d+2}), \quad V = (\mathbf{v}_1, \dots, \mathbf{v}_{m+n-2d+2}), \end{aligned} \quad (38)$$

with $\mathbf{u}_j \in \mathbf{R}^{m+n-d+1}$, $\mathbf{v}_j \in \mathbf{R}^{m+n-2d+2}$, and $\Sigma = \text{diag}(\sigma_1, \dots, \sigma_{m+n-2d+2})$ denotes the diagonal matrix whose the j -th diagonal element is σ_j . Note that U and V are orthogonal matrices. Then, by a property of the SVD ([9, Theorem 3.3]), the smallest singular value $\sigma_{m+n-2d+2}$ gives the minimum distance of the image of the unit sphere $S^{(m+n-2d+2)-1}$, given as

$$S^{(m+n-2d+2)-1} = \{\mathbf{x} \in \mathbf{R}^{m+n-2d+2} \mid \|\mathbf{x}\|_2 = 1\},$$

by N_{d-1} , represented as

$$N_{d-1} \cdot S^{(m+n-2d+2)-1} = \{N_{d-1}\mathbf{x} \mid \mathbf{x} \in \mathbf{R}^{m+n-2d+2}, \|\mathbf{x}\|_2 = 1\},$$

from the origin, along with $\sigma_{m+n-2d+2}\mathbf{u}_{m+n-2d+2}$ as its coordinates. By (38), we have

$$N_{d-1} \cdot \mathbf{v}_{m+n-2d+2} = \sigma_{m+n-2d+2}\mathbf{u}_{m+n-2d+2},$$

thus $\mathbf{v}_{m+n-2d+2}$ represents the coefficients of $A(x)$ and $B(x)$: let

$$\begin{aligned}\mathbf{v}_{m+n-2d+2} &= {}^t(\bar{a}_{n-d}, \dots, \bar{a}_0, \bar{b}_{m-d}, \dots, \bar{b}_0), \\ \bar{A}(x) &= \bar{a}_{n-d}x^{n-d} + \dots + \bar{a}_0x^0, \\ \bar{B}(x) &= \bar{b}_{m-d}x^{m-d} + \dots + \bar{b}_0x^0.\end{aligned}$$

Then, $\bar{A}(x)$ and $\bar{B}(x)$ give the least norm of $AF + BG$ satisfying $\|A\|_2^2 + \|B\|_2^2 = 1$ by putting $A(x) = \bar{A}(x)$ and $B(x) = \bar{B}(x)$.

Therefore, we admit the coefficients of F , G , \bar{A} and \bar{B} as the initial values of the iterations as

$$\mathbf{x}_0 = (f_m, \dots, f_0, g_n, \dots, g_0, \bar{a}_{n-d}, \dots, \bar{a}_0, \bar{b}_{m-d}, \dots, \bar{b}_0). \quad (39)$$

4.3.2. The Complex Coefficient Case

In the complex case, we calculate the SVD of $N = \begin{pmatrix} N_1 & -N_2 \\ N_2 & N_1 \end{pmatrix}$ in (19) as

$$\begin{aligned}N &= U \Sigma {}^tV, \quad U = (\mathbf{u}_1, \dots, \mathbf{u}_{2(m+n-2d+2)}), \\ \Sigma &= \text{diag}(\sigma_1, \dots, \sigma_{2(m+n-2d+2)}), \quad V = (\mathbf{v}_1, \dots, \mathbf{v}_{2(m+n-2d+2)}),\end{aligned} \quad (40)$$

where $\mathbf{u}_j \in \mathbf{R}^{2(m+n-d+1)}$, $\mathbf{v}_j \in \mathbf{R}^{2(m+n-2d+2)}$, and U and V are orthogonal matrices. Then, as in the case of the real coefficients, the smallest singular value $\sigma_{2(m+n-2d+2)}$ gives the minimum distance of the image of the unit sphere $\mathbf{S}^{2(m+n-2d+2)-1}$, given as

$$\mathbf{S}^{2(m+n-2d+2)-1} = \{\mathbf{x} \in \mathbf{R}^{2(m+n-2d+2)} \mid \|\mathbf{x}\|_2 = 1\},$$

by N , represented as

$$N \cdot \mathbf{S}^{2(m+n-2d+2)-1} = \{N\mathbf{x} \mid \mathbf{x} \in \mathbf{R}^{2(m+n-2d+2)}, \|\mathbf{x}\|_2 = 1\},$$

from the origin, along with $\sigma_{2(m+n-2d+2)}\mathbf{u}_{2(m+n-2d+2)}$ as its coordinates. By (40), we have

$$N \cdot \mathbf{v}_{2(m+n-2d+2)} = \sigma_{2(m+n-2d+2)}\mathbf{u}_{2(m+n-2d+2)},$$

thus $\mathbf{v}_{2(m+n-2d+2)}$ represents the coefficients of $A(x)$ and $B(x)$: let

$$\begin{aligned}\mathbf{v}_{2(m+n-2d+2)} &= {}^t(\bar{a}_{n-d,1}, \dots, \bar{a}_{0,1}, \bar{b}_{m-d,1}, \dots, \bar{b}_{0,1}, \bar{a}_{n-d,2}, \dots, \bar{a}_{0,2}, \bar{b}_{m-d,2}, \dots, \bar{b}_{0,2}), \\ \bar{A}(x) &= (\bar{a}_{n-d,1} + \bar{a}_{n-d,2}\mathbf{i})x^{n-d} + \dots + (\bar{a}_{0,1} + \bar{a}_{0,2}\mathbf{i})x^0, \\ \bar{B}(x) &= (\bar{b}_{m-d,1} + \bar{b}_{m-d,2}\mathbf{i})x^{m-d} + \dots + (\bar{b}_{0,1} + \bar{b}_{0,2}\mathbf{i})x^0.\end{aligned}$$

Then, $\bar{A}(x)$ and $\bar{B}(x)$ give the least norm of $AF + BG$ satisfying $\|A\|_2^2 + \|B\|_2^2 = 1$ by putting $A(x) = \bar{A}(x)$ and $B(x) = \bar{B}(x)$ in (14).

Therefore, we admit the coefficients of F , G , \bar{A} and \bar{B} as the initial values of the iterations as

$$\mathbf{x}_0 = (f_{m,1}, \dots, f_{0,1}, g_{n,1}, \dots, g_{0,1}, f_{m,2}, \dots, f_{0,2}, g_{n,2}, \dots, g_{0,2}, \bar{a}_{n-d,1}, \dots, \bar{a}_{0,1}, \bar{b}_{m-d,1}, \dots, \bar{b}_{0,1}, \bar{a}_{n-d,2}, \dots, \bar{a}_{0,2}, \bar{b}_{m-d,2}, \dots, \bar{b}_{0,2}). \quad (41)$$

4.4. Regarding the Minimization Problem as the Minimum Distance (Least Squares) Problem

Since we have the object function f as in (12) or (23) in the case of the real or the complex coefficients, respectively, we have $\nabla f(\mathbf{x}) = 2\mathbf{v}_R$, where

$$\mathbf{v}_R = {}^t(x_1 - f_m, \dots, x_{m+1} - f_0, x_{m+2} - g_n, \dots, x_{m+n+2} - g_0, 0, \dots, 0), \quad (42)$$

in the case of the real coefficients, or $\nabla f(\mathbf{x}) = 2\mathbf{v}_C$, where

$$\mathbf{v}_C = {}^t(x_1 - f_{m,1}, \dots, x_{m+1} - f_{0,1}, x_{m+2} - g_{n,1}, \dots, x_{m+n+2} - g_{0,1}, x_{m+n+3} - f_{m,2}, \dots, x_{2m+n+3} - f_{0,2}, x_{2m+n+4} - g_{n,2}, \dots, x_{2(m+n+2)} - g_{0,2}, 0, \dots, 0), \quad (43)$$

in the case of the complex coefficients, respectively. However, we can regard our problem as finding a point $\mathbf{x} \in V_q$ which has the minimum distance to the initial point \mathbf{x}_0 with respect to the (x_1, \dots, x_{m+n+2}) -coordinates in the case of the real coefficients or the $(x_1, \dots, x_{2(m+n+2)})$ -coordinates in the case of the complex coefficients, respectively, which correspond to the coefficients in $F(x)$ and $G(x)$. Therefore, in the gradient projection method at $\mathbf{x} \in V_q$, the projection of $-\nabla f(\mathbf{x})$ in (28) should be the projection of \mathbf{v}_R in the case of the real coefficients, or \mathbf{v}_C in the case of the complex coefficients, respectively, onto $T_{\mathbf{x}}$, where \mathbf{v}_R and \mathbf{v}_C are as in (42) and (43), respectively. These changes are equivalent to changing the objective function as $\tilde{f}(\mathbf{x}) = \frac{1}{2}f(\mathbf{x})$ then solving the minimization problem of $\tilde{f}(\mathbf{x})$, subject to $\mathbf{q}(\mathbf{x}) = \mathbf{0}$.

4.5. Calculating the Actual GCD and Correcting the Deformed Polynomials

After successful end of the iterations in Algorithms 1 or 2, we obtain the coefficients of $\tilde{F}(x)$, $\tilde{G}(x)$, $A(x)$ and $B(x)$ satisfying (4) with $A(x)$ and $B(x)$ are relatively prime. Then, we need to compute the actual GCD $H(x)$ of $\tilde{F}(x)$ and $\tilde{G}(x)$. Although H can be calculated as the quotient of \tilde{F} divided by B or \tilde{G} divided by A , naive polynomial division may cause numerical errors in the coefficient. Thus, we calculate the coefficients of H by the so-called least squares division ([31]), followed by correcting the coefficients in \tilde{F} and \tilde{G} by using the calculated H , as follows.

4.5.1. Calculating Candidates for the GCD in the Real Coefficient Case

For polynomials \tilde{F} , \tilde{G} , A and B represented as in (5) and H represented as

$$H(x) = h_d x^d + \dots + h_0 x^0,$$

solve the equations $HB = \tilde{F}$ and $HA = \tilde{G}$ with respect to H as solving the least squares problems of linear systems

$$C_d(A)^t(h_d, \dots, h_0) = {}^t(\tilde{g}_n, \dots, \tilde{g}_0), \quad (44)$$

$$C_d(B)^t(h_d, \dots, h_0) = {}^t(\tilde{f}_m, \dots, \tilde{f}_0), \quad (45)$$

respectively. Let $H_1(x), H_2(x) \in \mathbf{R}[x]$ be the candidates for the GCD whose coefficients are calculated as the least squares solutions of (44) and (45), respectively.

4.5.2. Calculating Candidates for the GCD in the Complex Coefficient Case

For polynomials \tilde{F} , \tilde{G} , A and B represented as in (14) and H represented as

$$H(x) = (h_{d,1} + h_{d,2}\mathbf{i})x^d + \dots + (h_{0,1} + h_{0,2}\mathbf{i})x^0,$$

solve the equations $HB = \tilde{F}$ and $HA = \tilde{G}$ with respect to H as solving the least squares problems of linear systems

$$C_d(A)^t(h_{d,1} + h_{d,2}\mathbf{i}, \dots, h_{0,1} + h_{0,2}\mathbf{i}) = {}^t(\tilde{g}_{n,1} + \tilde{g}_{n,2}\mathbf{i}, \dots, \tilde{g}_{0,1} + \tilde{g}_{0,2}\mathbf{i}), \quad (46)$$

$$C_d(B)^t(h_{d,1} + h_{d,2}\mathbf{i}, \dots, h_{0,1} + h_{0,2}\mathbf{i}) = {}^t(\tilde{f}_{m,1} + \tilde{f}_{m,2}\mathbf{i}, \dots, \tilde{f}_{0,1} + \tilde{f}_{0,2}\mathbf{i}), \quad (47)$$

respectively. Then, we transfer the linear systems (46) and (47), as follows. For (47), let us express the matrices and vectors as the sum of the real and the imaginary part of which, respectively, as

$$\begin{aligned} C_d(B) &= B_1 + \mathbf{i}B_2, \\ {}^t(h_{d,1} + h_{d,2}\mathbf{i}, \dots, h_{0,1} + h_{0,2}\mathbf{i}) &= \mathbf{h}_1 + \mathbf{i}\mathbf{h}_2, \\ {}^t(\tilde{f}_{m,1} + \tilde{f}_{m,2}\mathbf{i}, \dots, \tilde{f}_{0,1} + \tilde{f}_{0,2}\mathbf{i}) &= \mathbf{f}_1 + \mathbf{i}\mathbf{f}_2. \end{aligned}$$

Then, (45) is expressed as

$$(B_1 + \mathbf{i}B_2)(\mathbf{h}_1 + \mathbf{i}\mathbf{h}_2) = (\mathbf{f}_1 + \mathbf{i}\mathbf{f}_2). \quad (48)$$

By equating the real and the imaginary parts in Eq. (48), respectively, we have

$$(B_1\mathbf{h}_1 - B_2\mathbf{h}_2) = \mathbf{f}_1, \quad (B_1\mathbf{h}_2 + B_2\mathbf{h}_1) = \mathbf{f}_2,$$

or

$$\begin{pmatrix} B_1 & -B_2 \\ B_2 & B_1 \end{pmatrix} \begin{pmatrix} \mathbf{h}_1 \\ \mathbf{h}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{pmatrix}. \quad (49)$$

Thus, we can calculate the coefficients of $H(x)$ by solving the real least squares problem (49). We can solve (46) similarly. Let $H_1(x), H_2(x) \in \mathbf{C}[x]$ be the candidates for the GCD whose coefficients are calculated as the least squares solutions of (46) and (47), respectively.

4.5.3. Choosing the GCD and Calculating the Deformed Polynomials

Let $H_1(x), H_2(x) \in \mathbf{C}[x]$ be the candidates for the GCD calculated as in the above. Then, for $i = 1, 2$, calculate the norms of the residues as

$$r_i = \|\tilde{F} - H_i B\|_2^2 + \|\tilde{G} - H_i A\|_2^2,$$

respectively, and set the GCD $H(x)$ be $H_i(x)$ giving the minimum value of r_i .

Finally, for the chosen $H(x)$, correct the coefficients of $\tilde{F}(x)$ and $\tilde{G}(x)$ as

$$\tilde{F}(x) = H(x) \cdot B(x), \quad \tilde{G}(x) = H(x) \cdot A(x),$$

respectively.

4.6. The Algorithm

Summarizing the above, the algorithm for calculating approximate GCD becomes as follows.

Algorithm 3 (GPGCD: Approximate GCD by the Gradient-Projection Method).

- Inputs:
 - $F(x), G(x) \in \mathbf{R}[x]$ or $\mathbf{C}[x]$ with $\deg(F) \geq \deg(G) > 0$,
 - $d \in \mathbf{N}$: the degree of approximate GCD with $d \leq \deg(G)$,
 - $\varepsilon > 0$: a threshold for terminating iteration in the gradient-projection method,
 - $u \in \mathbf{N}$: an upper bound for the number of iterations permitted in the gradient-projection method.
- Outputs: $\tilde{F}(x), \tilde{G}(x), H(x) \in \mathbf{R}[x]$ or $\mathbf{C}[x]$ such that \tilde{F} and \tilde{G} are deformations of F and G , respectively, whose GCD is equal to H with $\deg(H) = d$.

Step 1 [Setting the initial values] As the discussions in Section 4.3, set the initial values \mathbf{x}_0 as in (39) in the case of the real coefficients, or (41) in the case of the complex coefficients, respectively.

Step 2 [Iteration] As the discussions in Section 4.4, solve the minimization problem of $\bar{f}(\mathbf{x}) = \frac{1}{2}f(\mathbf{x})$, subject to $\mathbf{q}(\mathbf{x}) = \mathbf{0}$, with $f(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ as in (12) and (13) in the case of the real coefficients, or in (23) and (24) in the case of the complex coefficients, respectively. Apply Algorithm 1 or 2 for the minimization: repeat iterations until the search direction \mathbf{d}_k (as in (28) in the gradient-projection method or in (31) in a modified Newton method, respectively) satisfies $\|\mathbf{d}_k\|_2 < \varepsilon$, or the number of iteration reaches its upper bound u .

Step 3 [Construction of \tilde{F} , \tilde{G} and H] As the discussions in Section 4.5, construct the GCD $H(x)$ and correct the coefficients of $\tilde{F}(x)$ and $\tilde{G}(x)$. Then, return $\tilde{F}(x)$, $\tilde{G}(x)$ and $H(x)$. If Step 2 did not end with the number of iterations less than u , report it to the user.

4.7. Preserving Monicity

While Algorithm 3 permits changing the leading coefficients for calculating $\tilde{F}(x)$ and $\tilde{G}(x)$, we can also give an algorithm restricting inputs $F(x)$ and $G(x)$ and outputs $\tilde{F}(x)$ and $\tilde{G}(x)$ to be monic as follows.

4.7.1. The Real Coefficient Case

Let $\tilde{F}(x)$ and $\tilde{G}(x)$ be represented as in (5) with $\tilde{f}_m = \tilde{g}_n = 1$, then, by Eq. (7), we have $b_{m-d} = -a_{n-d}$. Thus, we eliminate the variables \tilde{f}_m , \tilde{g}_n and b_{m-d} , which cause the following changes.

Changes on the Subresultant Matrix. By eliminating the variables as in the above, we see that Eq. (7) is equivalent to

$$N'_{d-1}(\tilde{F}, \tilde{G}) \cdot {}^t(a_{n-d}, \dots, a_0, b_{m-d-1}, \dots, b_0) = \mathbf{0},$$

where $N'_{d-1}(\tilde{F}, \tilde{G})$ is defined as

$$N'_{d-1}(\tilde{F}, \tilde{G}) = \begin{pmatrix} \tilde{f}_{m-1} - \tilde{g}_{n-1} & 1 & & & 1 & & \\ & \vdots & \tilde{f}_{m-1} & \ddots & & \tilde{g}_{n-1} & \ddots \\ \tilde{f}_0 - \tilde{g}_{n-m} & \vdots & \ddots & & 1 & \vdots & \ddots & 1 \\ & \tilde{f}_0 & & \tilde{f}_{m-1} & \tilde{g}_0 & & \tilde{g}_{n-1} & \\ & & \ddots & \vdots & & \ddots & \vdots & \\ & & & \tilde{f}_0 & & & \tilde{g}_0 & \end{pmatrix},$$

with (in the first column) $\tilde{g}_j = 0$ for $j < 0$, by subtracting the first column by the $(n-d+1)$ -th column, then deleting the first row and the $(n-d+1)$ -th column (corresponding to the b_{m-d} term) in $N_{d-1}(\tilde{F}, \tilde{G})$.

Changes on the Settings in the Minimization Problem. In solving the minimization problem, we substitute the variables

$$(\tilde{f}_{m-1}, \dots, \tilde{f}_0, \tilde{g}_{n-1}, \dots, \tilde{g}_0, a_{n-d}, \dots, a_0, b_{m-d-1}, \dots, b_0)$$

as $\mathbf{x} = (x_1, \dots, x_{2(m+n-d)+1})$, instead of (11). As a consequence, in contrast to (12), the objective function $f(\mathbf{x})$ becomes as

$$\begin{aligned} f(\mathbf{x}) = & (x_1 - \tilde{f}_{m-1})^2 + \dots + (x_m - \tilde{f}_0)^2 \\ & + (x_{m+1} - \tilde{g}_{n-1})^2 + \dots + (x_{m+n} - \tilde{g}_0)^2. \end{aligned} \quad (50)$$

Also, in contrast to (9) and (10), the constraints $\mathbf{q}(\mathbf{x})$ become as

$$\begin{aligned} q_0 &= 2a_{n-d}^2 + a_{n-d-1}^2 \dots + a_0^2 + b_{m-d-1}^2 + \dots + b_0^2 - 1 = 0, \\ q_1 &= (\tilde{f}_{m-1} - \tilde{g}_{n-1})a_{n-d} + a_{n-d-1} + b_{m-d-1} = 0, \\ &\vdots \\ q_{m+n-d} &= \tilde{f}_0 a_0 + \tilde{g}_0 b_0 = 0. \end{aligned} \quad (51)$$

Changes on the Initial Values. Let $N'_{d-1} = U \Sigma^t V$ be the SVD of $N'_{d-1}(F, G)$, with

$$V = (\mathbf{v}_1, \dots, \mathbf{v}_{m+n-2d-1}),$$

$$\mathbf{v}_{m+n-2d-1} = {}^t(\bar{a}_{n-d}, \dots, \bar{a}_0, \bar{b}_{m-d-1}, \dots, \bar{b}_0).$$

Then, in contrast to (39), the initial values become as

$$\mathbf{x}_0 = (f_{m-1}, \dots, f_0, g_{n-1}, \dots, g_0, \bar{a}_{n-d}, \dots, \bar{a}_0, \bar{b}_{m-d-1}, \dots, \bar{b}_0). \quad (52)$$

The Algorithm. Summarizing discussions in the above, for preserving $\tilde{F}(x)$ and $\tilde{G}(x)$ to be monic, we modify Algorithm 3 as follows.

Algorithm 4 (GPGCD preserving monicity, with real coefficients). Change Steps 1 and 2 in Algorithm 3 as follows.

Step 1 [Setting the initial values] Set the initial values \mathbf{x}_0 as in (52).

Step 2 [Iteration] Solve the minimization problem of $\bar{f}(\mathbf{x}) = \frac{1}{2}(\mathbf{x})$, subject to $\mathbf{q}(\mathbf{x}) = \mathbf{0}$, with $f(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ defined as in (50) and (51), respectively, as Step 2 in Algorithm 3.

4.7.2. The Complex Coefficient Case

Let $\tilde{F}(x)$ and $\tilde{G}(x)$ be represented as in (14) with $\tilde{f}_{m,1} = \tilde{g}_{n,1} = 1$ and $\tilde{f}_{m,2} = \tilde{g}_{n,2} = 0$, then, by Eq. (7), we have $b_{m-d,j} = -a_{n-d,j}$ for $j \in \{1, 2\}$. Thus, for $j \in \{1, 2\}$, we eliminate the variables $\tilde{f}_{m,j}$, $\tilde{g}_{n,j}$ and $b_{m-d,j}$, which cause the following changes.

Changes on the Subresultant Matrix. By eliminating the variables as in the above, we see that Eq. (16) is equivalent to

$$\begin{pmatrix} (\tilde{f}_{m-1,1} - \tilde{g}_{n-1,1}) + (\tilde{f}_{m-1,2} - \tilde{g}_{n-1,2})\mathbf{i} & 1 & & & \\ & \vdots & \tilde{f}_{m-1,1} + \tilde{f}_{m-1,2}\mathbf{i} & \ddots & \\ & & \vdots & \ddots & 1 \\ (\tilde{f}_{0,1} - \tilde{g}_{n-m,1}) + (\tilde{f}_{0,2} - \tilde{g}_{n-m,2})\mathbf{i} & & \tilde{f}_{0,1} + \tilde{f}_{0,2}\mathbf{i} & \tilde{f}_{m-1,1} + \tilde{f}_{m-1,2}\mathbf{i} & \\ & & & \ddots & \vdots \\ & & & & \tilde{f}_{0,1} + \tilde{f}_{0,2}\mathbf{i} \end{pmatrix} \begin{pmatrix} 1 & & & & \\ \tilde{g}_{n-1,1} + \tilde{g}_{n-1,2}\mathbf{i} & \ddots & & & \\ \vdots & \ddots & 1 & & \\ \tilde{g}_{0,1} + \tilde{g}_{0,2}\mathbf{i} & & \tilde{g}_{n-1,1} + \tilde{g}_{n-1,2}\mathbf{i} & & \\ & & \vdots & & \\ & & \tilde{g}_{0,1} + \tilde{g}_{0,2}\mathbf{i} & & \end{pmatrix} \begin{pmatrix} a_{n-d,1} + a_{n-d,2}\mathbf{i} \\ \vdots \\ a_{0,1} + a_{0,2}\mathbf{i} \\ b_{m-d-1,1} + b_{m-d-1,2}\mathbf{i} \\ \vdots \\ b_{0,1} + b_{0,2}\mathbf{i} \end{pmatrix} = \mathbf{0}, \quad (53)$$

with (in the first column of the matrix in the left-hand-side) $\tilde{g}_{i,j} = 0$ for $i < 0$ and $j \in \{1, 2\}$, in which the matrix in the left-hand-side is obtained by subtracting the first column by the $(n-d+1)$ -th column, then deleting the first row and the $(n-d+1)$ -th column (corresponding to the $b_{m-d,1} + b_{m-d,2}\mathbf{i}$ term) in the corresponding matrix in (16). Then, Eq. (17) becomes as

$$(N'_1 + N'_2\mathbf{i})(\mathbf{v}'_1 + \mathbf{v}'_2\mathbf{i}) = \mathbf{0},$$

with

$$\begin{aligned} N'_1 &= \begin{pmatrix} \tilde{f}_{m-1,1} - \tilde{g}_{n-1,1} & 1 & & 1 \\ \vdots & \tilde{f}_{m-1,1} & \ddots & \tilde{g}_{n-1,1} & \ddots \\ \tilde{f}_{0,1} - \tilde{g}_{n-m,1} & \vdots & \ddots & 1 & \vdots & \ddots & 1 \\ & \tilde{f}_{0,1} & \tilde{f}_{m-1,1} & \tilde{g}_{0,1} & \tilde{g}_{n-1,1} \\ & & \ddots & \vdots & \ddots & \vdots \\ & & & \tilde{f}_{0,1} & & \tilde{g}_{0,1} \end{pmatrix}, \\ N'_2 &= \begin{pmatrix} \tilde{f}_{m-1,2} - \tilde{g}_{n-1,2} & 1 & & 1 \\ \vdots & \tilde{f}_{m-1,2} & \ddots & \tilde{g}_{n-1,2} & \ddots \\ \tilde{f}_{0,2} - \tilde{g}_{n-m,2} & \vdots & \ddots & 1 & \vdots & \ddots & 1 \\ & \tilde{f}_{0,2} & \tilde{f}_{m-1,2} & \tilde{g}_{0,2} & \tilde{g}_{n-1,2} \\ & & \ddots & \vdots & \ddots & \vdots \\ & & & \tilde{f}_{0,2} & & \tilde{g}_{0,2} \end{pmatrix}, \\ \mathbf{v}'_1 &= {}^t(a_{n-d,1}, \dots, a_{0,1}, b_{m-d-1,1}, \dots, b_{0,1}), \\ \mathbf{v}'_2 &= {}^t(a_{n-d,2}, \dots, a_{0,2}, b_{m-d-1,2}, \dots, b_{0,2}). \end{aligned} \quad (54)$$

Changes on the Settings in the Minimization Problem. In solving the minimization problem, we substitute the variables

$$(\tilde{f}_{m-1,1}, \dots, \tilde{f}_{0,1}, \tilde{g}_{n-1,1}, \dots, \tilde{g}_{0,1}, \tilde{f}_{m-1,2}, \dots, \tilde{f}_{0,2}, \tilde{g}_{n-1,2}, \dots, \tilde{g}_{0,2}, \\ a_{n-d,1}, \dots, a_{0,1}, b_{m-d-1,1}, \dots, b_{0,1}, a_{n-d,2}, \dots, a_{0,2}, b_{m-d-1,2}, \dots, b_{0,2})$$

as $\mathbf{x} = (x_1, \dots, x_{4(m+n-d+1)})$, instead of (22). As a consequence, in contrast to (23), the objective function $f(\mathbf{x})$ becomes as

$$\begin{aligned} f(\mathbf{x}) &= (x_1 - f_{m-1,1})^2 + \dots + (x_m - f_{0,1})^2 \\ &\quad + (x_{m+1} - g_{n-1,1})^2 + \dots + (x_{m+n} - g_{0,1})^2 \\ &\quad + (x_{m+n+1} - f_{m-1,2})^2 + \dots + (x_{2m+n} - f_{0,2})^2 \\ &\quad + (x_{2m+n+1} - g_{n-1,2})^2 + \dots + (x_{2(m+n)} - g_{0,2})^2. \end{aligned} \quad (55)$$

The constraints becomes as follows. Now, Eq. (19) becomes as

$$\begin{pmatrix} N'_1 & -N'_2 \\ N'_2 & N'_1 \end{pmatrix} \begin{pmatrix} \mathbf{v}'_1 \\ \mathbf{v}'_2 \end{pmatrix} = \mathbf{0}, \quad (56)$$

with $N'_1, N'_2, \mathbf{v}'_1$ and \mathbf{v}'_2 are defined as in (54). Furthermore, the constraint for the coefficients in $A(x)$ and $B(x)$ as in (20) now becomes as

$$\begin{aligned} \|A(x)\|_2^2 + \|B(x)\|_2^2 &= (2a_{n-d,1}^2 + \cdots + a_{0,1}^2) + (b_{m-d-1,1}^2 + \cdots + b_{0,1}^2) \\ &+ (2a_{n-d,2}^2 + \cdots + a_{0,2}^2) + (b_{m-d-1,2}^2 + \cdots + b_{0,2}^2) - 1 = 0. \end{aligned} \quad (57)$$

Then, by the same way we have constructed (21), we put (56) and (57) together as

$$\begin{pmatrix} {}^t\mathbf{v}'_1 & {}^t\mathbf{v}'_2 & a_{n-d,1}^2 + a_{n-d,2}^2 - 1 \\ N'_1 & -N'_2 & \mathbf{0} \\ N'_2 & N'_1 & \mathbf{0} \end{pmatrix} \begin{pmatrix} \mathbf{v}'_1 \\ \mathbf{v}'_2 \\ 1 \end{pmatrix} = \mathbf{0}, \quad (58)$$

and we obtain the constraint $\mathbf{q}(\mathbf{x}) = \mathbf{0}$ as

$$\mathbf{q}(\mathbf{x}) = {}^t(q_1(\mathbf{x}), \dots, q_{2(m+n-d)+1}(\mathbf{x})) = \mathbf{0}, \quad (59)$$

where $q_j(\mathbf{x})$ corresponds to the j -th row of matrix-vector product in (58).

Changes on the Initial Values. Let $N' = U \Sigma {}^tV$ be the SVD of $N' = \begin{pmatrix} N'_1 & -N'_2 \\ N'_2 & N'_1 \end{pmatrix}$, with

$$\begin{aligned} V &= (\mathbf{v}_1, \dots, \mathbf{v}_{2(m+n-2d+1)}), \\ \mathbf{v}_{2(m+n-2d+1)} &= {}^t(\bar{a}_{n-d,1}, \dots, \bar{a}_{0,1}, \bar{b}_{m-d-1,1}, \dots, \bar{b}_{0,1}, \\ &\quad \bar{a}_{n-d,2}, \dots, \bar{a}_{0,2}, \bar{b}_{m-d-1,2}, \dots, \bar{b}_{0,2}). \end{aligned}$$

Then, in contrast to (41), the initial value becomes as

$$\begin{aligned} \mathbf{x}_0 &= (f_{m-1,1}, \dots, f_{0,1}, g_{n-1,1}, \dots, g_{0,1}, f_{m-1,2}, \dots, f_{0,2}, g_{n-1,2}, \dots, g_{0,2}, \\ &\quad \bar{a}_{n-d,1}, \dots, \bar{a}_{0,1}, \bar{b}_{m-d-1,1}, \dots, \bar{b}_{0,1}, \bar{a}_{n-d,2}, \dots, \bar{a}_{0,2}, \bar{b}_{m-d-1,2}, \dots, \bar{b}_{0,2}). \end{aligned} \quad (60)$$

The Algorithm. Summarizing discussions in the above, for preserving $\tilde{F}(x)$ and $\tilde{G}(x)$ to be monic, we modify Algorithm 3 as follows.

Algorithm 5 (GPGCD preserving monicity, with complex coefficients). Change Steps 1 and 2 in Algorithm 3 as follows.

Step 1 [Setting the initial values] Set the initial values \mathbf{x}_0 as in (60).

Step 2 [Iteration] Solve the minimization problem of $\bar{f}(\mathbf{x}) = \frac{1}{2}(\mathbf{x})$, subject to $\mathbf{q}(\mathbf{x}) = \mathbf{0}$, with $f(\mathbf{x})$ and $\mathbf{q}(\mathbf{x})$ defined as in (55) and (59), respectively, as Step 2 in Algorithm 3.

4.7.3. Running Time Analysis

We give an analysis for running time of Algorithm 3 with employing the modified Newton method.

In Step 1, we set the initial values by the SVD. Since the dimension of subresultant matrix is $O(m+n-d)$, running time in this step becomes $O((m+n-d)^3)$.

In Step 2, we estimate running time just for one iteration, since the number of iterations for convergence of solution may vary depending on the given problem. This step essentially depends on solving the linear system (30) with the Jacobian matrix $J_q(\mathbf{x}_k)$ defined as in (32) for the real coefficient case or in (34) for the complex coefficient case. In both cases, dimension of $J_q(\mathbf{x}_k)$ is $O(m + n - d)$, thus we can estimate running time for solving the linear system (30) as $O((m + n - d)^3)$.

Step 3 depends on calculating the least square solution of the linear system as in Section 4.5 whose running time becomes as $O((m + n - d)^3)$.

As a consequence, we can estimate running time of Algorithm 3 as $O((m + n - d)^3)$ times the number of iterations for finding a GCD.

4.8. Examples

Now we show examples of Algorithm 3 in the case of the real coefficients (more comprehensive experiments are presented in the next section).

Note that, for the minimization method, we have employed a modified Newton method (Algorithm 2). Computations in Example 1 have been executed on a computer algebra system Mathematica 6 with hardware floating-point arithmetic, while those in Examples 2 and 3 have been executed on another computer algebra system Maple 15 with `Digits=10`.

Example 1. This example is given by Karmarkar and Lakshman [15], followed by Kaltofen et al. [14]. Let $F(x), G(x) \in \mathbf{R}[x]$ be

$$\begin{aligned} F(x) &= x^2 - 6x + 5 = (x - 1)(x - 5), \\ G(x) &= x^2 - 6.3x + 5.72 = (x - 1.1)(x - 5.2), \end{aligned}$$

and find $\tilde{F}(x), \tilde{G}(x) \in \mathbf{R}[x]$ which have the GCD of degree 1, namely $\tilde{F}(x)$ and $\tilde{G}(x)$ have one common zero.

Case 1: The leading coefficient can be perturbed. Applying Algorithm 3 to F and G , with $d = 1$ and $\varepsilon = 1.0 \times 10^{-8}$, after 7 iterations, we obtain the polynomials \tilde{F} and \tilde{G} as

$$\begin{aligned} \tilde{F}(x) &= 0.985006x^2 - 6.00294x + 4.99942, \\ \tilde{G}(x) &= 1.01495x^2 - 6.29707x + 5.72058, \end{aligned}$$

with perturbations as $\sqrt{\|\tilde{F} - F\|_2^2 + \|\tilde{G} - G\|_2^2} = 0.0215941$ and the common zero of $\tilde{F}(x)$ and $\tilde{G}(x)$ as $x = 5.09890419203$. In Kaltofen et al. [14], the calculated perturbations obtained is $\sqrt{0.0004663} = 0.021594$ with the common zero as $x = 5.09890429$. Karmarkar and Lakshman [15] only give an example without perturbations on the leading coefficients.

Case 2: The leading coefficient cannot be perturbed. Applying Algorithm 3 (preserving monicity) with the same arguments as in Case 1, after 7 iterations, we obtain the polynomials \tilde{F} and \tilde{G} as

$$\begin{aligned} \tilde{F}(x) &= x^2 - 6.07504x + 4.98528, \\ \tilde{G}(x) &= x^2 - 6.22218x + 5.73527, \end{aligned}$$

with perturbations as $\sqrt{\|\tilde{F} - F\|_2^2 + \|\tilde{G} - G\|_2^2} = 0.110164$. And the common zero of $\tilde{F}(x)$ and $\tilde{G}(x)$ as $x = 5.0969464650$. In Kaltofen et al. [14], the calculated perturbations obtained is $\sqrt{0.01213604583} = 0.110164$ with the common zero as $x = 5.0969478$. In Karmarkar and Lakshman [15], the calculated perturbations obtained is $\sqrt{0.01213605293} = 0.110164$ with the common zero as $x = 5.096939087$.

The next examples, originally by Sanuki and Sasaki [21], are ill-conditioned ones with the small or large leading coefficient GCD.

Example 2 (A small leading coefficient problem [21, Example 4]). Let $F(x)$ and $G(x)$ be

$$\begin{aligned} F(x) &= (x^4 + x^2 + x + 1)(0.001x^2 + x + 1), \\ G(x) &= (x^3 + x^2 + x + 1)(0.001x^2 + x + 1). \end{aligned}$$

Applying Algorithm 3 to F and G , with $d = 2$ and $\varepsilon = 1.0 \times 10^{-8}$, after 1 iteration, we obtain the polynomials \tilde{F} , \tilde{G} and H as

$$\begin{aligned} \tilde{F}(x) &\simeq F(x), \quad \tilde{G}(x) \simeq G(x), \\ H(x) &= 0.001x^2 + 0.9999999936x + 0.9999999936, \end{aligned}$$

with $\sqrt{\|\tilde{F} - F\|_2^2 + \|\tilde{G} - G\|_2^2} = 8.485281374 \times 10^{-12}$.

Example 3 (A large leading coefficient problem [21, Example 5]). Let $F(x)$ and $G(x)$ be

$$\begin{aligned} F(x) &= (x^6 - 0.00001(0.8x^5 + 3x^4 - 4x^3 - 4x^2 - 5x + 1)) \cdot C(x), \\ G(x) &= (x^5 + x^4 + x^3 - 0.1x^2 + 1) \cdot C(x), \end{aligned}$$

with $C(x) = x^2 + 0.001$. Applying Algorithm 3 to F and G , with $d = 2$ and $\varepsilon = 1.0 \times 10^{-8}$, after 1 iteration, we obtain the polynomials \tilde{F} , \tilde{G} and H as

$$\begin{aligned} \tilde{F}(x) &\simeq F(x), \quad \tilde{G}(x) \simeq G(x), \\ H(x) &= x^2 + 1.548794164 \times 10^{-16}x + 0.001, \end{aligned}$$

with $\sqrt{\|\tilde{F} - F\|_2^2 + \|\tilde{G} - G\|_2^2} = 1.735004369 \times 10^{-14}$.

5. Experiments

We have implemented our GPGCD method (Algorithm 3) on a computer algebra system Maple² and carried out the following tests:

²The implementation is available at Project Hosting on Google Code [27].

1. (Section 5.1) Comparison of performance of the gradient-projection method (Algorithm 1) and the modified Newton method (Algorithm 2) on randomly generated polynomials with approximate GCD,
2. (Section 5.2) Comparison of performance of the GPGCD method with a method based on the structured total least norm (STLN) method by Kaltofen et al. [13] and the UVGCD method by Zeng [31] on large sets of randomly-generated polynomials with approximate GCD,
3. (Section 5.3) Comparison of performance of the GPGCD method with the STLN-based method and the UVGCD method on ill-conditioned polynomials and other test cases by Zeng [31] and Bini and Boito [2].

Note that, in Test 2, we have tested both the cases of the real and the complex coefficients, while, in the other tests, we have tested only the case of the real coefficients.

In Tests 1 and 2, we have generated random polynomials with GCD then added noise, as follows. First, we have generated a pair of monic polynomials $F_0(x)$ and $G_0(x)$ of degrees m and n , respectively, with the GCD of degree d . The GCD and the prime parts of degrees $m - d$ and $n - d$ are generated as monic polynomials and with random coefficients $c \in [-10, 10]$ of floating-point numbers. For noise, we have generated a pair of polynomials $F_N(x)$ and $G_N(x)$ of degrees $m - 1$ and $n - 1$, respectively, with random coefficients as the same as for $F_0(x)$ and $G_0(x)$. Then, we have defined a pair of test polynomials $F(x)$ and $G(x)$ as

$$F(x) = F_0(x) + \frac{e_F}{\|F_N(x)\|_2} F_N(x), \quad G(x) = G_0(x) + \frac{e_G}{\|G_N(x)\|_2} G_N(x),$$

respectively, scaling the noise such that the 2-norm of the noise for F and G is equal to e_F and e_G , respectively. In the present test, we set $e_F = e_G = 0.1$. (See also the notes in Section 5.2.)

The tests have been carried out on Intel Core2 Duo Mobile Processor T7400 (in Apple MacBook “Mid-2007” model) at 2.16 GHz with RAM 2GB, under Mac OS X 10.6. All the tests have been carried out on Maple 15 with `Digits=15` executing hardware floating-point arithmetic.

5.1. Test 1: Comparison of the Gradient-Projection Method and the Modified Newton Method

In this test, we have compared performance of the gradient-projection method (Algorithm 1) and a modified Newton method (Algorithm 2), only in the case of the real coefficients. For every example, we have generated one random test polynomial as in the above, and we have applied Algorithm 3 (preserving monicity) with $u = 100$ and $\varepsilon = 1.0 \times 10^{-8}$.

Table 1 shows the result of the test: m and n denotes the degree of a tested pair F and G , respectively, and d denotes the degree of approximate GCD; “Perturbation” is the perturbation of the perturbed polynomials from the initial inputs, calculated as

$$\sqrt{\|\tilde{F} - F\|_2^2 + \|\tilde{G} - G\|_2^2}, \quad (61)$$

Ex.	m, n	d	Perturbation (61)	#Iterations		Time (sec.)	
				Alg. 1	Alg. 2	Alg. 1	Alg. 2
1	10, 10	5	$4.25e-2$	3	4	0.10	0.04
2	20, 20	10	$6.86e-2$	3	4	0.17	0.11
3	40, 40	20	$6.80e-2$	4	5	0.61	0.16
4	60, 60	30	$7.24e-2$	3	4	0.69	0.23
5	80, 80	40	$5.06e-2$	3	4	1.41	0.41
6	100, 100	50	$7.26e-2$	3	4	2.21	0.76

Table 1: Test results comparing the gradient-projection method and the modified Newton method; see Section 5.1 for details.

where “ aeb ” with a and b as numbers denotes $a \times 10^b$; “#Iterations” is the number of iterations; “Time” is computing time in seconds. The columns with “Alg. 1” and “Alg. 2” are the data for Algorithm 1 (the gradient-projection method) and Algorithm 2 (the modified Newton method), respectively. Note that, the “Perturbation” is a single column since both algorithms give almost the same values in each examples.

We see that, in all the test cases, the number of iterations of the gradient-projection method (Algorithm 1) is equal to 3 or 4, which is smaller than that of the modified Newton method (Algorithm 2) which is equal to 4 or 5. However, an iteration in Algorithm 1 includes solving a linear system at least twice: once in the *projection step* (Step 2) and at least once in the *restoration step* (Step 3); whereas an iteration in Algorithm 2 includes that only once. Thus, total number of solving a linear system in Algorithm 2 is about a half of that in Algorithm 1. Furthermore, computing time shows that the modified Newton method runs approximately twice as fast as the gradient projection method. Therefore, we adopt Algorithm 2 as the method of minimization in the GPGCD method (Algorithm 3).

5.2. Test 2: Tests on Large Sets of Randomly-generated Polynomials

In this test, we have compared Algorithm 3 with a method based on the structured total least norm (STLN) method ([13]) and the UVGCD method ([30]), using their implementation for the *Maple*, in the both cases of the real and the complex coefficients. In our implementation of Algorithm 3, we have chosen the modified Newton method (Algorithm 2) for minimization. In the STLN-based method, we have used their procedure `R_con_mulpoly` and `C_con_mulpoly`, which calculates the approximate GCD of several polynomials in $\mathbf{R}[x]$ and $\mathbf{C}[x]$, respectively. In the UVGCD method, we have used their procedure `uvgcd` for calculating approximate GCD of polynomials in $\mathbf{R}[x]$ and $\mathbf{C}[x]$.

Note that, in this test, we have defined test polynomials satisfying another requirement: to make sure that the input polynomials $F(x)$ and $G(x)$ do not have a GCD of degree exceeding d , we have adopted only those satisfying that the smallest singular value of the d -th subresultant matrix $N_d(F, G)$ (see (3)) is

Ex.	m, n	d	Perturbation (61)			Time (sec.)			#Iterations	
			STLN	UVGCD	GPGCD	STLN	UVGCD	GPGCD	STLN	GPGCD
1	10, 10	5	$5.64e-2$	$1.79e-1$	$5.64e-2$	0.38	0.64	0.04	4.46	4.50
2	20, 20	10	$6.22e-2$	$1.85e-1$	$6.22e-2$	1.16	0.86	0.06	4.40	4.40
3	30, 30	15	$6.65e-2$	$1.87e-1$	$6.65e-2$	2.43	1.34	0.10	4.37	4.46
4	40, 40	20	$6.48e-2$	$1.96e-1$	$6.48e-2$	4.05	2.09	0.13	4.11	4.15
5	50, 50	25	$6.91e-2$	$1.91e-1$	$6.91e-2$	6.30	3.34	0.19	4.03	4.16
6	60, 60	30	$6.75e-2$	$1.94e-1$	$6.75e-2$	9.09	4.37	0.26	4.00	4.18
7	70, 70	35	$6.89e-2$	$2.08e-1$	$6.89e-2$	12.47	5.71	0.35	3.96	4.13
8	80, 80	40	$6.78e-2$	$1.91e-1$	$6.78e-2$	16.95	7.95	0.44	3.16	4.11
9	90, 90	45	$6.92e-2$	$1.95e-1$	$6.92e-2$	22.09	10.20	0.57	3.96	4.10
10	100, 100	50	$6.98e-2$	$1.95e-1$	$6.98e-2$	27.48	13.02	0.69	3.88	4.09

Table 2: Test results for large sets of polynomials with approximate GCD, in the case of the real coefficients; see Section 5.2 for details.

larger than or equal to 1.³

For every example, we have generated 100 random test polynomials as in the above. In executing Algorithm 3, we have set $u = 200$ and $\varepsilon = 1.0 \times 10^{-8}$; in `R_con_mulpoly` and `C_con_mulpoly`, we have set the tolerance $e = 1.0 \times 10^{-8}$; in `uvgcd`, we have set the initial tolerance $\delta = 1.0 \times 10^{-2}$ and have changed it until we have obtained an approximate GCD of desired degree.

Tables 2 and 3 show the results of the test in the case of the real and the complex coefficients, respectively: m and n denotes the degree of a pair F and G , respectively, and d denotes the degree of approximate GCD. The columns with “STLN” are the data for the STLN-based method; “UVGCD” are the data for the UVGCD method; “GPGCD” are the data for the GPGCD method (Algorithm 3). “Perturbation”, “#Iterations” and “Time” are the same as those in Table 1, respectively. (Note that computing time for the UVGCD method does not include the time for “try and error” calculations by changing the tolerance δ : it is just for successful calculations.)

We see that the average of magnitude of perturbations by the GPGCD method is as small as that by the STLN-based method, which is approximately one-tenth as large as that by the UVGCD method. For computing time, the GPGCD method calculates approximate GCD very efficiently, faster than the STLN-based method by approximately from 10 to 30 times and the UVGCD method by approximately from 6 to 10 times.

Remark 3. In this experiment, we have compared our implementation designed

³Our previous test results ([26, Section 5.2]) have shown that there were test cases (input polynomials with the real coefficients) in which the GPGCD method was not able to calculate an approximate GCD with sufficiently small magnitude of perturbations. After thorough investigation, we have found that such input polynomials accidentally have an approximate GCD of degree exceeding d . Thus, in the present test, we have defined totally new test polynomials satisfying the above requirement, then none of such phenomena have been observed with the test.

Ex.	m, n	d	Perturbation (61)			Time (sec.)			#Iterations	
			STLN	UVGCD	GPGCD	STLN	UVGCD	GPGCD	STLN	GPGCD
1	10, 10	5	$5.92e-2$	$1.54e-1$	$5.92e-2$	1.58	0.64	0.11	4.50	4.46
2	20, 20	10	$6.40e-2$	$1.41e-1$	$6.40e-2$	5.34	1.31	0.20	4.30	4.30
3	30, 30	15	$6.63e-2$	$1.40e-1$	$6.63e-2$	11.63	2.12	0.35	4.21	4.24
4	40, 40	20	$6.61e-2$	$1.34e-1$	$6.61e-2$	21.57	3.51	0.55	4.15	4.13
5	50, 50	25	$6.86e-2$	$1.48e-1$	$6.86e-2$	34.23	5.03	0.83	4.06	4.10
6	60, 60	30	$6.86e-2$	$1.51e-1$	$6.86e-2$	50.40	7.39	1.16	4.02	4.05
7	70, 70	35	$6.94e-2$	$1.41e-1$	$6.94e-2$	69.54	10.31	1.56	3.93	4.05
8	80, 80	40	$6.85e-2$	$1.44e-1$	$6.85e-2$	93.77	14.01	2.07	3.91	4.07
9	90, 90	45	$6.84e-2$	$1.52e-1$	$6.84e-2$	122.97	18.30	2.65	3.90	4.04
10	100, 100	50	$6.94e-2$	$1.65e-1$	$6.94e-2$	157.02	23.72	3.37	3.86	4.04

Table 3: Test results for large sets of polynomials with approximate GCD, in the case of the complex coefficients; see Section 5.2 for details.

for problems of two univariate polynomials against the implementation of the STLN-based method designed for multivariate multi-polynomial problems with additional linear coefficient constraints. Kaltofen [12] has reported that they have tested their implementation for just two univariate polynomials with real coefficients ([14]) on an example similar to ours with degree 100 and GCD degree 50, and it took (on a ThinkPad of 1.8 GHz with RAM 1GB) 2 iterations and 9 seconds. This result will give the reader some idea on efficiency of our method.

5.3. Test 3: Tests for Ill-conditioned Polynomials and Other Cases

In this test, we have compared Algorithm 3 with the STLN-based method ([13]) and the UVGCD method ([30]) on some ill-conditioned polynomials and other test cases by Zeng [31] and Bini and Boito [2], as follows.

Note that we give the degree of approximate GCD in the STLN-based method and the GPGCD method, while we give the tolerance δ then the algorithm estimates the degree of approximate GCD in the UVGCD method. Also note that, in some tests in this section, we have measured the relative error of approximate GCD from the given GCD (63) instead of the magnitude of perturbation (61) because, in such cases, we have given test polynomials with predefined (approximate) GCD and have intended to observe “nearness” of the calculated approximate GCD from the predefined one.

Throughout the tables in this section, the columns with “STLN”, “UVGCD”, “GPGCD”, “Perturbation”, and “Time” the same as those in the above, respectively.

Example 4. An example of ill-conditioned polynomial by Zeng [31, Test 1]. Let n be an even positive number and $k = n/2$, and define $p_n = u_n v_n$ and

n	Relative error of GCD (63)		
	STLN	UVGCD	GPGCD
6	$1.04e-14$	$4.60e-15$	$3.68e-15$
8	$3.98e-13$	$7.90e-13$	$4.30e-13$
10	$1.08e-10$	$7.89e-12$	$1.08e-10$
12	$2.87e-10$	$2.95e-11$	$2.94e-10$
14	$3.10e-9$	$3.65e-10$	$3.14e-9$
16	$6.22e-9$ (*1)	$3.83e-10$	$8.00e-9$
18	$1.38e-6$ (*1)	$9.68e-9$	$1.36e-6$
20	$6.95e-6$ (*1)	$1.21e-8$	$7.11e-6$ (*2)

Table 4: Test results for test polynomials (62). See Example 4 for details.

$q_n = u_n w_n$, where

$$\begin{aligned}
u_n &= \prod_{j=1}^k [(x - r_1 \alpha_j)^2 + r_1^2 \beta_j^2], & v_n &= \prod_{j=1}^k [(x - r_2 \alpha_j)^2 + r_2^2 \beta_j^2], \\
w_n &= \prod_{j=k+1}^n [(x - r_1 \alpha_j)^2 + r_1^2 \beta_j^2], & \alpha_j &= \cos \frac{j\pi}{n}, \quad \beta_j = \sin \frac{j\pi}{n},
\end{aligned} \tag{62}$$

for $r_1 = 0.5$ and $r_2 = 1.5$. The zeros of p_n and q_n lie on the circles of radius r_1 and r_2 . We had the test for $n = 6, \dots, 20$ increased by 2.

Table 4 shows the result of the test. “Relative error of GCD” is calculated by

$$\frac{\|\bar{u}_n(x) - u_n(x)\|_2}{\|u_n(x)\|_2}, \tag{63}$$

where u_n is predefined GCD as shown in (62) and \bar{u}_n is approximate GCD. In the table, (*1) indicates that the STLN-based method did not converge within 50 times of iterations which is a built-in threshold; (*2) indicates that the GPGCD method did not converge within 100 times of iterations. We see that, in the GPGCD method as well as in the STLN-based method, the number of iterations increases and the accuracy of calculated approximate GCD decreases as n increases. On the other hand, the UVGCD method has better accuracy of approximate GCD for large n .

Example 5. Another example of ill-conditioned polynomial by Zeng [31, Test 2]. Let

$$p(x) = \prod_{j=1}^{10} (x - x_j), \quad q(x) = \prod_{j=1}^{10} (x - x_j + 10^{-j}), \quad x_j = (-1)^j (j/2), \tag{64}$$

The zeros of q have decreasing distances as $0.1, 0.01, \dots$, from those of p . We have tried to calculate an approximate GCD of degree d from 1 to 10 increased by 1.

d	Perturbation (61)	
	STLN	GPGCD
1	$5.17e-1$ (*1)	$3.21e3$
2	$6.95e-4$ (*1)	$3.06e0$
3	$1.97e-5$	$1.26e0$
4	$2.89e-6$	$2.25e-1$
5	$5.28e-5$	$4.75e-1$
6	$2.15e-3$	$2.16e-3$
7	$8.34e-2$	$8.34e-2$
8	$2.04e0$	$2.04e0$
9	$4.70e1$	$4.70e1$
10	$7.73e2$	$7.73e2$

Table 5: Test results for test polynomials (64) with the STLN-based method and the GPGCD method. See Example 5 for details.

Tables 5 and 6 show the result of the test. In this test, we have measured perturbation (61) since p and q are pairwise relatively prime in a rigorous sense. Note that we have put the results for the UVGCD method in Table 6, separated from those for the GPGCD and the STLN-based methods in Table 5, because we have given the tolerance δ to obtain approximate GCD in the UVGCD method, while we have given the degree d in the GPGCD and the STLN-based methods. In Table 5, (*1) indicates that the STLN-based method did not converge within 50 times of iterations which is a built-in threshold.

We see that, for $d \geq 6$, all the methods find approximate GCD with similar magnitude of perturbations. However, for smaller value of d , the UVGCD method finds approximate GCD with considerably smaller magnitude of perturbations than those in the other methods, followed by the STLN-based method.

Example 6. An example with GCDs of large degree by Zeng [31, Test 3]. Let

$$\begin{aligned}
p_n &= u_n v, & q_n &= u_n w, \\
v(x) &= \sum_{j=0}^3 x^j, & w(x) &= \sum_{j=0}^3 (-x)^j,
\end{aligned} \tag{65}$$

where $u_n(x)$ is a GCD defined as a polynomial of degree n whose coefficients are random integers in the range $[-5, 5]$ and $v(x)$ and $w(x)$ are fixed cofactors.

Table 7 shows the result of the test by measuring relative error of approximate GCD (63). In this test, we have also measured computing time because the difference of it became large among the methods for large degree of approximate GCD. We see that the UVGCD method calculates approximate GCD with the best accuracy, followed by the STLN-based method and the GPGCD method. On the other hand, the GPGCD method is more efficient than the other methods.

UVGCD		
δ	d	Perturbation (61)
$1.0e-11$	1	$8.02e-10$
$1.0e-10$	2	$3.27e-8$
$1.0e-9$	3	$6.03e-7$
$1.0e-8$	4	$1.99e-5$
$1.0e-7$	5	$3.45e-4$
$1.0e-6$	5	$3.45e-4$
$1.0e-5$	6	$9.61e-3$
$1.0e-4$	7	$1.79e-1$
$1.0e-3$	8	$3.18e0$
$1.0e-2$	8	$3.18e0$
$1.0e-1$	9	$5.00e1$
$1.0e0$	10	$8.40e2$

Table 6: Test results for test polynomials (64) with the UVGCD method. See Example 5 for details.

n	Relative error of GCD (63)			Time (sec.)		
	STLN	UVGCD	GPGCD	STLN	UVGCD	GPGCD
50	$1.60e-15$	$1.04e-16$	$2.63e-15$	1.77	0.22	0.04
100	$1.16e-15$	$1.59e-16$	$4.41e-15$	8.17	0.31	0.06
200	$1.14e-15$	$1.06e-16$	$1.23e-14$	45.09	0.83	0.12
500	$1.35e-15$	$1.37e-16$	$1.84e-14$	552.09	3.39	0.64
1000	$1.42e-15$	$1.69e-16$	$5.30e-14$	4318.38	18.66	3.27

Table 7: Test results for test polynomials (65). See Example 6 for details.

Example 7. An example with multiple zeros of high multiplicities by Bini and Boito [2, Example 4.5]. Let

$$u_k(x) = (x^3 + 3x - 1)(x - 1)^k, \quad v_k(x) = u'(x), \quad (66)$$

for positive integer k . Note that the GCD of $u_k(x)$ and $v_k(x)$ is $w_k(x) = (x - 1)^{k-1}$.

Table 8 shows the result of the test. In the table, as in Example 4, (*1) indicates that the STLN-based method did not converge within 50 times of iterations which is a built-in threshold; (*2) indicates that the GPGCD method did not converge within 100 times of iterations.

We see that, in the GPGCD method as well as in the STLN-based method, the number of iterations increases and the accuracy of calculated approximate GCD decreases for $k = 35$ and 45 . On the other hand, the UVGCD method calculates approximate GCD accurately for large k .

Example 8. Another example with multiple zeros of high multiplicities by Zeng

k	Relative error of GCD (63)		
	STLN	UVGCD	GPGCD
15	$2.35e-13$	$3.08e-15$	$1.86e-12$
25	$1.64e-11$	$1.13e-14$	$6.67e-11$
35	$3.79e-10$ (*1)	$8.02e-15$	$3.58e-9$ (*2)
45	$4.23e-8$ (*1)	$1.13e-14$	$1.78e-7$ (*2)

Table 8: Test results for test polynomials (66). See Example 7 for details.

$[m_1, m_2, m_3, m_4]$	Relative error of GCD (63)		
	STLN	UVGCD	GPGCD
$[2, 1, 1, 0]$	$1.11e-13$	$9.42e-16$	$2.83e-13$
$[3, 2, 1, 0]$	$7.33e-13$	$3.31e-15$	$8.23e-12$
$[4, 3, 2, 1]$	$2.35e-9$	$2.95e-13$	$2.68e-9$
$[5, 3, 2, 1]$	$1.89e-8$	$3.38e-12$	$5.56e-9$
$[9, 6, 4, 2]$	$4.72e-8$ (*1)	$5.31e-11$	$6.05e-8$ (*2)
$[20, 14, 10, 5]$	$5.06e-1$ (*1)	$3.13e-10$	$9.98e-1$ (*2)
$[80, 60, 40, 20]$	$1.0e0$ (*1)	$1.08e-3$	$1.0e0$ (*2)
$[100, 60, 40, 20]$	$1.0e0$ (*1)	$2.16e-4$	N/A (*3)

Table 9: Test results for test polynomials (67). See Example 8 for details.

[31, Test 6]. Let

$$\begin{aligned}
p_{[m_1, m_2, m_3, m_4]}(x) &= (x-1)^{m_1}(x-2)^{m_2}(x-3)^{m_3}(x-4)^{m_4}, \\
q_{[m_1, m_2, m_3, m_4]}(x) &= \frac{d}{dx} p_{[m_1, m_2, m_3, m_4]}(x),
\end{aligned} \tag{67}$$

for nonnegative integers m_1, \dots, m_4 . Note that the GCD of $p_{[m_1, m_2, m_3, m_4]}(x)$ and $q_{[m_1, m_2, m_3, m_4]}(x)$ is $(x-1)^{m'_1}(x-2)^{m'_2}(x-3)^{m'_3}(x-4)^{m'_4}$ with $m'_j = \max\{m_j - 1, 0\}$ for $j = 1, \dots, 4$.

Table 9 shows the result of the test. In the table, as in Examples 4 and 7, (*1) indicates that the STLN-based method did not converge within 50 times of iterations which is a built-in threshold; (*2) indicates that the GPGCD method did not converge within 100 times of iterations. Furthermore, (*3) indicates that the GPGCD method stopped abnormally because the solution of a linear system with the coefficient matrix (the Jacobian matrix) as shown in (32) became unexpectedly large.

We see that, in the GPGCD method as well as in the STLN-based method, the number of iterations increases and the accuracy of calculated approximate GCD becomes almost meaningless for inputs of large degree. On the other hand, the UVGCD method is quite stable (in the sense of convergence of the algorithm) and more accurate for calculating approximate GCD for those inputs.

6. Concluding Remarks

We have proposed an iterative method, based on the modified Newton method which is a generalization of the gradient-projection method, for calculating approximate GCD of univariate polynomials with the real or the complex coefficients.

Our experiments comparing the GPGCD method with the STLN-based method and the UVGCD method have discovered advantages and disadvantages of these methods, as follows. In the case that input polynomials already have exact or approximate GCD, then the UVGCD method calculates the approximate GCD with the best accuracy and relatively fast convergence among them. On the other hand, in the case that the magnitude of “noise” is larger, then the magnitude of perturbations calculated by the GPGCD method or the STLN-based method is smaller than that calculated by the UVGCD method. Furthermore, in such cases, the GPGCD method has shown significantly better performance over the other methods in its speed, by approximately up to 30 times for the STLN-based method and 10 times for the UVGCD method, which seems to be sufficiently practical. Other examples have shown that the GPGCD method properly calculates approximate GCD with small or large leading coefficient.

Our result have shown that, in contrast to the STLN-based methods which uses *structure preserving* feature for matrix computations, our simple method can achieve accurate and efficient computation as or more than theirs in calculating approximate GCDs in many examples. On the other hand, our result have also shown that our method is less accurate than the UVGCD method especially in the case the given polynomials lie sufficiently close to polynomials that have a GCD in a rigorous sense. These results suggest that there are some opportunities for improvements of accuracy and/or efficiency in calculating approximate GCDs with optimization strategies.

For the future research, the followings are of interest.

- Convergence analysis of the minimizations: showing global convergence of local method is difficult in general (see e.g. Blum et al. [3]), as the original paper on the modified Newton method ([24]) only shows its stability by observing whether the Jacobian matrix of the constraint at a local minimal point has full-rank or not. However, it may be possible to analyze local convergence property depending on condition on the initial point and/or local minimal point. (See also Remarks 1 and 2).
- Improvements on the efficiency: time complexity of our method depends on the minimization, or solving a system of linear equations in each iteration. Thus, analyzing the structure of matrices might improve the efficiency in solving a linear system.
- Comparison with other methods (approaches) for approximate GCD: from various points of view such as accuracy, stability, efficiency, and so on, comparison of our methods with other methods will reveal advantages and drawbacks of our method in more detail.

Other topics, such as generalization of our method to several input polynomials, are also among our next problems, some of which are currently under our investigation ([28]).

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